Math 165

Final

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Name: Answer Key 2

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Part A

1. (20 pts)

[10 points] (a) Find the solution to the differential equation

$$3x - e^y\sqrt{x^2+1}\frac{dy}{dx} = 0$$

which satisfies the initial condition $y(0) = 0$.

$$3x = e^y\sqrt{x^2+1}\,\frac{dy}{dx}$$

$$\int \frac{3x}{\sqrt{x^2+1}}\,dx = \int e^y\,dy + C$$

$u = x^2+1$

$du = 2x\,dx$

$$\frac{3}{2}\int u^{-1/2}\,du = e^y + C$$

$$\frac{3}{\cancel{2}}\,\frac{u^{1/2}}{\cancel{1/2}} = e^y + C$$

$$3(x^2+1)^{1/2} = e^y + C$$

$y(0)=0 \rightarrow$

$$3 = 1 + C \rightarrow C = 2$$

$$3(x^2+1)^{1/2} = e^y + 2$$

$$y = \ln(3\sqrt{x^2+1} - 2)$$

Answer: $y(x) =$ $\ln(3\sqrt{x^2+1} - 2)$

[10 points] (b) Find the solution to the differential equation

$$y' + y\cos x = 6\cos x$$

which satisfies the initial condition $y(0) = 7$.

$$P(x) = \cos x$$

$$e^{\int P(x)dx} = e^{\sin x}$$

$$(ye^{\sin x})' = 6\cos x\, e^{\sin x}$$

$$ye^{\sin x} = \int \underbrace{6\cos x\, e}_{\substack{u = \sin x \\ du = \cos x dx}}{}^{\sin x} dx + C$$

$$= \int 6e^{u} du + C$$

$$ye^{\sin x} = 6e^{\sin x} + C$$

$$y = 6 + Ce^{-\sin x}$$

$y(0) = 7$ $\quad 7 = 6 + Ce^{0} \longrightarrow C = 1$

$$\therefore \quad y = 6 + e^{-\sin x}$$

Answer: $y(x) =$ $6 + e^{-\sin x}$

2. (14 pts)

Use Gauss-Jordan row reduction to find the inverse of the matrix

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 2 & -3 & 3 \\ 1 & -1 & 1 \end{bmatrix}$$

if it exists.

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 2 & -3 & 3 & 0 & 1 & 0 \\ 1 & -1 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\downarrow$$

$$\left[\begin{array}{ccc|ccc} \textcircled{1} & -1 & 2 & 1 & 0 & 0 \\ 0 & \textcircled{+1} & +1 & +2 & -1 & 0 \\ 0 & 0 & -1 & -1 & 0 & 1 \end{array}\right]$$

$$\downarrow$$

$$\left[\begin{array}{ccc|ccc} \textcircled{1} & 0 & 3 & 3 & -1 & 0 \\ 0 & \textcircled{1} & 1 & 2 & -1 & 0 \\ 0 & 0 & \textcircled{1} & 1 & 0 & -1 \end{array}\right]$$

$$\downarrow$$

$$\left[\begin{array}{ccc|ccc} \textcircled{1} & 0 & 0 & 0 & -1 & 3 \\ 0 & \textcircled{1} & 0 & 1 & -1 & 1 \\ 0 & 0 & \textcircled{1} & 1 & 0 & -1 \end{array}\right]$$

Answer: $A^{-1} =$ $\begin{bmatrix} 0 & -1 & 3 \\ 1 & -1 & 1 \\ 1 & 0 & -1 \end{bmatrix}$

3. (15 pts) Consider a system of linear equations $A\mathbf{x} = \mathbf{b}$ where A is an $m \times n$ matrix. Let $r = rank(A)$. In each of the following cases, what can be said about the number of solutions to the system?

1. If $m = n$ and A is invertible, then the system

   (a) is inconsistent.

   (b) has a unique solution.

   (c) has infinitely many solutions.

   (d) Further information is necessary to determine which of (a), (b) or (c) occur.

   Answer: (Write one of the letters 'a,b,c' or 'd'): b

2. If $m = n$ and A is not invertible, then the system

   (a) is inconsistent.

   (b) has a unique solution.

   (c) has infinitely many solutions.

   (d) Further information is necessary to determine which of (a), (b) or (c) occur.

   Answer: (Write one of the letters 'a,b,c' or 'd'): d

3. If $m = n$, $r < n$ and $\mathbf{b} = \mathbf{0}$ then the system

   (a) is inconsistent.

   (b) has a unique solution.

   (c) has infinitely many solutions.

   (d) Further information is necessary to determine which of (a), (b) or (c) occur.

   Answer: (Write one of the letters 'a,b,c' or 'd'): c

4. If $m < n$ and $\mathbf{b} = \mathbf{0}$, then the system

   (a) is inconsistent.

   (b) has a unique solution.

   (c) has infinitely many solutions.

   (d) Further information is necessary to determine which of (a), (b) or (c) occur.

   Answer: (Write one of the letters 'a,b,c' or 'd'): c

5. If $m > n$, $r = n$ and $\mathbf{b} = \mathbf{0}$, then the system

   (a) is inconsistent.

   (b) has a unique solution.

   (c) has infinitely many solutions.

   (d) Further information is necessary to determine which of (a), (b) or (c) occur.

   Answer: (Write one of the letters 'a,b,c' or 'd'): b

4. (20 pts)

[10 points] (a) Find the determinant of

$$M = \begin{pmatrix} 3 & 5 & -1 \\ 2 & 2 & 7 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\det M = -1 \begin{vmatrix} 3 & -1 \\ 2 & 7 \end{vmatrix} = -(21+2) = -23$$

Answer: $\det(M) =$ -23

[10 points] (b) Suppose A is a 4×4 matrix with $\det(A) = 3$ and B is obtained from A by subtracting 2 times row 3 from row 2. Then:

(i) $\det(2A) =$ $2^4 \cdot 3 = 48$

(ii) $\det(A^T) =$ 3

(iii) $\det(A^{-1}) =$ $1/3$

(iv) $\det(A^3) =$ 27

(v) $\det(B) =$ 3

5. **(16 pts)** Determine which of the following subsets of P_3 are **subspaces** of P_3. (P_3 is the vector space of real polynomials of degree 3 or less.) For each subset, **circle** NO if it is not a subspace and list a subspace property that fails for this subset in the provided slot. **Circle** YES if it is a subspace and in this case find and enter the dimension of this subspace in the slot provided.

(a) $S_1 = \{p(t) \in P_3 \mid p'(t) - p(t) + 3 = 0 \text{ for all } t\}$

[✓ NO] it is not a subspace. A subspace property that fails to hold is lacks zero

YES it is a subspace and its dimension is ____

(b) $S_2 = \{p(t) \in P_3 \mid p(-t) = -p(t) \text{ for all } t\}$

$$a+bt+ct^2+dt^3 = a-bt+ct^2-dt^3$$

$$\rightarrow b=d=0$$

NO it is not a subspace. A subspace property that fails to hold is ________

[✓ YES] it is a subspace and its dimension is 2

(c) $S_3 = \{p(t) \in P_3 \mid p(0) = 1\}$

[✓ NO] it is not a subspace. A subspace property that fails to hold is lacks zero

YES it is a subspace and its dimension is ____

(d) $S_4 = \{p(t) \in P_3 \mid p'(2) = p(1)\}$

$$\cancel{b} + 2c(2) + 3d(2)^2 = a + \cancel{b} + c + d$$

$$a = 3c + 11d$$

NO it is not a subspace. A subspace property that fails to hold is ________

[✓ YES] it is a subspace and its dimension is 3

6. (15 pts)

The reduced row echelon form of

$$A = \begin{pmatrix} 3 & 1 & -6 & 0 & 3 \\ 2 & 1 & -4 & 0 & -1 \\ 3 & 0 & -6 & 1 & 12 \end{pmatrix}$$

is

$$U = \begin{pmatrix} 1 & 0 & -2 & 0 & 4 \\ 0 & 1 & 0 & 0 & -9 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

[3 points] (a) The rank of A is

Answer: Rank of A is: 3

Answer: Reason given for answer above: Has 3 pivots in RREF

[3 points] (b) The nullity of A is

Answer: Nullity of A is: 2

Answer: Reason given for answer above: rank + nullity = 5

So 3 + nullity = 5

[3 points] (c) List a set of basis vectors for the column space of A.

Answer: Basis for column space of A is: $\begin{bmatrix} 3 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

Answer: Reason given for answer above:

Columns of A corr. to pivot columns in RREF

[3 points] (d) List a set of basis vectors for the null space of A.

Answer: Basis for nullspace of A is: $\begin{bmatrix} 2 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -4 \\ 9 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 2x_3 - 4x_5 \\ 9x_5 \\ x_3 \\ 0 \\ x_5 \end{bmatrix}$$

Answer: Reason given for answer above: x_3, x_5 free

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 2x_3 - 4x_5 \\ 9x_5 \\ x_3 \\ 0 \\ x_5 \end{bmatrix} = x_3 \begin{bmatrix} 2 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -4 \\ 9 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

[3 points] (e) Give an example of a nontrivial linear dependency amongst the columns of A.

Answer: Example of nontrivial linear dependency is:

$2(\text{col}1) + (\text{col}3) = \vec{0}$

Part B

1. (17 pts) For the differential equation

$$(D^2+1)^2(D+2)y = e^t,$$

[7 points] (a) Find the general solution y_c to its associated homogeneous differential equation $(D^2+1)^2(D+2)y = 0$. (Use real functions in your final solution, i.e. do not leave answers in terms of complex exponentials if they occur.)

$$y_c = Ae^{-2t} + B\cos t + C\sin t + Dt\cos t + Et\sin t$$

Answer: $y_c =$

$$Ae^{-2t} + B\cos t + C\sin t + Dt\cos t + Et\sin t$$

[7 points](b) Find a particular solution y_p to the differential equation.

$$y_p = Fe^t$$

$$2^2(3)\overset{Fe^t}{\text{[illegible]}} = \text{[illegible]}e^t$$

$$12F = 1 \longrightarrow F = 1/12$$

Answer: $y_p =$ $\frac{1}{12}e^t$

[3 points](c) Determine the general solution to the differential equation.

Answer: $y =$

$$Ae^{-2t} + B\cos t + C\sin t + Dt\cos t + Et\sin t + \frac{1}{12}e^t$$

2. **(16 pts)** Consider the 3×3 matrix

$$A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 1 & -1 & 2 \end{bmatrix}$$

[4 points] (a) Determine the eigenvalues of A.

Answer: Eigenvalues are: 3, 2, 2

[8 points] (b) Determine the eigenspaces corresponding to each of the eigenvalues of A. In your answer make sure to label so that it can be determined which eigenspace belongs to which eigenvalue.

2-eigenspace: $\text{Null}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & -1 & 0 \end{bmatrix} = \text{Null}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \text{Span}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

3-eigenspace: $\text{Null}\begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 1 & -1 & -1 \end{bmatrix} = \text{Null}\begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \text{Null}\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \text{Span}\left(\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}\right)$

Answer: Eigenspaces are:

2-eigen $= \text{Span}\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right)$, 3-eigen $= \text{Span}\left(\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}\right)$

[4 points] (c) Determine if A is defective. Justify your answer.

Write final answer. A is defective [✓ YES] or NO: YES

Answer: Explanation: Only 2 LI eigenvectors for 3x3 matrix.

3. (17 pts) Solve the initial value problem

$$y'' + 6y' + 5y = 0$$

with $y(0) = 1, y'(0) = 0$.

$$x^2 + 6x + 5$$

[illegible]

$$= (x+5)(x+1)$$

$$y = Ce^{-5t} + De^{-t} \longrightarrow C + D = 1$$

$$y' = -5Ce^{-5t} - De^{-t} \longrightarrow -5C - D = 0$$

$$D = -5C$$

$$-4C = 1 \rightarrow \boxed{\begin{array}{l} C = -1/4 \\ D = 5/4 \end{array}}$$

Answer: $y(x) =$ $-\frac{1}{4}e^{-5t} + \frac{5}{4}e^{-t}$

4. **(15 pts)** A small mass m is attached to a wall with a horizontal spring with spring constant k. The floor the system lies on has friction coefficient c. The y-axis is perpendicular to the wall, pointing away from it, and the mass is confined to move along only this direction for this problem. As usual, we set $y = 0$ to be the rest position of the spring. Under these assumptions, with no further forces besides those of the spring and friction, the spring displacement y satisfies the following "simple harmonic oscillator" differential equation:

$$y'' + \frac{c}{m}y' + \frac{k}{m}y = 0.$$

where $m, k, c > 0$ and the independent variable is time t. Each part in the following scenarios is independent of each other part with different parameters:

[5 points] (a) The system is perturbed from its rest position with some initial displacement and velocity. Write down an inequality involving k, c, m under which oscillations will generally occur. Is the system called overdamped, critically damped or underdamped in this situation?

Answer: Inequality involving k, c, m under which the system will undergo oscillations: $c^2 < km$

Circle one option as answer: System is overdamped/critically damped/[✓ underdamped] in this situation.

[5 points] (b) Now assume c, k and m are such that the system is overdamped, write down the general solution to the system in this case (in terms of k, m, c, t and two general constants). What happens to the solutions when $t \to \infty$? Explain the last answer carefully.

Answer: $y(t) =$ $Ae^{\left(-\frac{c}{2m} - \frac{\sqrt{c^2-4k}}{2m}\right)t} + Be^{\left(-\frac{c}{2m} + \frac{\sqrt{c^2-4k}}{2m}\right)t}$

Answer: $\lim_{t\to\infty} y(t) =$ 0

Explanation of limit:

Both $-\frac{c}{2m} \pm \frac{\sqrt{c^2-4k}}{2m} < 0$ so the exponentials decay to 0 as $t \to \infty$

[5 points] (c) Now assume $c = 0, k = 9, m = 1$. What is the natural frequency of this system? If an external driving force $F_{ext} = 4cos(\omega t)$ is attached to the spring, for which driving frequency ω will the response be biggest (in the long run)? What is the name of the phenomena that motivates your last answer?

Natural frequency of the system is: $\sqrt{\frac{k}{m}} = \sqrt{9} = 3$

Response is biggest (in the long run) for $\omega =$ 3

Name of the phenomenon responsible for your last answer:

Resonance

5. (18 pts)

[9 points] (a) Suppose a system $\hat{x}' = \mathbb{A}\hat{x}$ where $\mathbb{A}$ is a 2×2 matrix has general solution

$$\hat{x} = C_1 e^{-t} \begin{bmatrix} 2 \\ 1 \end{bmatrix} + C_2 e^{5t} \begin{bmatrix} 3 \\ 2 \end{bmatrix}.$$

Find $\mathbb{A}$.

$\underbrace{A}_{2\times 2}$ has eigenvalues -1 and 5 with eigenvectors $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$

$$A = S\Lambda S^{-1} = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 0 & 5 \end{bmatrix}\begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 0 & 5 \end{bmatrix}\begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} -2 & 15 \\ -1 & 10 \end{bmatrix}\begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} -19 & 36 \\ -12 & 23 \end{bmatrix}$$

Answer: $\mathbb{A} =$ $\begin{bmatrix} -19 & 36 \\ -12 & 23 \end{bmatrix}$

[9 points] (b) Let $\mathbb{B}$ be a 2×2 real matrix which has eigenvalue $1 + 4i$ with corresponding eigenvector $\begin{bmatrix} 1 \\ 2 + 3i \end{bmatrix}$. Write down the general solution to $\hat{x}' = \mathbb{B}\hat{x}$ where the independent variable is time t. **Please make sure that the two basis solutions used in the final form of your general solution are real valued quantities.**

$$\hat{x} = A\underbrace{e^{(1+4i)t}\begin{bmatrix} 1 \\ 2+3i \end{bmatrix}}_{\hat{z}} + B\underbrace{e^{(1-4i)t}\begin{bmatrix} 1 \\ 2-3i \end{bmatrix}}_{\overline{\hat{z}}} = \underline{\underline{C\,\mathrm{Re}\,\hat{z} + D\,\mathrm{Im}\,\hat{z}}}$$

$$\hat{z} = e^t(\cos 4t + i\sin 4t)\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 0 \\ 3 \end{bmatrix}i\right)$$

$$= \underbrace{e^t\left(\cos(4t)\begin{bmatrix} 1 \\ 2 \end{bmatrix} - \sin(4t)\begin{bmatrix} 0 \\ 3 \end{bmatrix}\right)}_{\mathrm{Re}(\hat{z})} + i\, \underbrace{e^t\left(\cos(4t)\begin{bmatrix} 0 \\ 3 \end{bmatrix} + \sin(4t)\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right)}_{\mathrm{Im}(\hat{z})}$$

Answer: $\hat{x}(t) =$

$$e^t\left(C\begin{bmatrix} \cos(4t) \\ 2\cos(4t) - 3\sin(4t) \end{bmatrix} + D\begin{bmatrix} \sin(4t) \\ 3\cos(4t) + 2\sin(4t) \end{bmatrix}\right)$$

6. (17 pts) Consider the second order linear ODE:

$$y'' + 5y' + 6y = 0.$$

[5 points] (a) Rewrite this as a homogeneous linear system of first order ODEs: $\hat{x}' = \mathbb{A}\hat{x}$. Describe your choice of $\hat{x}$ and $\mathbb{A}$ explicitly.

$$\left.\begin{array}{l} X_1 = y \\ X_2 = y' \end{array}\right\} \to \begin{array}{l} X_1' = 0X_1 + 1X_2 \\ X_2' = y'' = -5y' - 6y = -5X_2 - 6X_1 \end{array}$$

Answer: $\hat{x} =$ $\begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$

Answer: $\mathbb{A} =$ $\begin{bmatrix} 0 & 1 \\ -6 & -5 \end{bmatrix}$

[9 points] (b) Find the eigenvalues and corresponding eigenvectors of your matrix $\mathbb{A}$ from part (a).

Char poly: $x^2 - (-5x) + (6)$

$= x^2 + 5x + 6$

$= (x+2)(x+3) \to$ eigenvalues $-2, -3$

(-2)-eigenspace $= \mathrm{Null}\left(\begin{bmatrix} 2 & 1 \\ -6 & -3 \end{bmatrix}\right) = \mathrm{Null}\left(\begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix}\right)$ $\left(\begin{array}{l} 2v_1 + v_2 = 0 \\ \to v_2 = -2v_1 \end{array}\right)$

$= \mathrm{Span}\left(\begin{bmatrix} 1 \\ -2 \end{bmatrix}\right)$

(-3)-eigenspace $= \mathrm{Null}\left(\begin{bmatrix} 3 & 1 \\ -6 & -2 \end{bmatrix}\right) = \mathrm{Null}\begin{bmatrix} 3 & 1 \\ 0 & 0 \end{bmatrix}$ $\left(\begin{array}{l} 3v_1 + v_2 = 0 \\ \to v_2 = -3v_1 \end{array}\right)$

$= \mathrm{Span}\left(\begin{bmatrix} 1 \\ -3 \end{bmatrix}\right)$

Eigenvalues and Eigenvectors:

$\lambda = -2$ $\begin{bmatrix} 1 \\ -2 \end{bmatrix}$ || $\lambda = -3$ $\begin{bmatrix} 1 \\ -3 \end{bmatrix}$

[3 points] (c) Write down the general solution to the system, i.e., the general solution for $\hat{x}$.

Answer: $\hat{x} =$ $C_1 e^{-2t}\begin{bmatrix} 1 \\ -2 \end{bmatrix} + C_2 e^{-3t}\begin{bmatrix} 1 \\ -3 \end{bmatrix}$