Pr Find the values of $\frac{\partial f}{\partial x}$ & $\frac{\partial f}{\partial y}$ at the point $(4,-5)$ if

$$f(x,y) = x^2 + 3xy + y - 1$$

Soln To find $\frac{\partial f}{\partial x}$, we treat y as a constant and differentiate wrt x.

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\left(x^2 + 3xy + y - 1\right) = 2x + 3y + 0 - 0$$

$$= 2x + 3y$$

$\Rightarrow$

$$\frac{\partial f}{\partial x}(4,-5) = 2.4 + 3.(-5) = -7.$$

and $\frac{\partial f}{\partial y}(4,-5) = 3.4 + 1 = 13.$

Pr Find $\frac{\partial f}{\partial y}$ if $f(x,y) = y \sin xy$

Soln

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(y \sin xy) = y \frac{\partial}{\partial y} \sin xy + (\sin xy)\frac{\partial}{\partial y} y$$

$$= xy \cos xy + \sin xy.$$

Pr Find $\frac{\partial z}{\partial x}$ if the eqn

$yz = \ln z + x + y$ defines z as a fnc of the two independent variables x & y and the partial derivative exist.

Soln We differentiate both sides of the eqn wrt x, holding y constant and treating z as a differentiable fnc of x.

$$\frac{\partial}{\partial x}(yz) = \frac{\partial}{\partial x}\ln z + \frac{\partial x}{\partial x} + \frac{\partial y}{\partial x}$$

$$y\frac{\partial z}{\partial x} = \frac{1}{z}\frac{\partial z}{\partial x} + 1 + 0$$

$$\left(y - \frac{1}{z}\right)\frac{\partial z}{\partial x} = 1$$

$$\Rightarrow \quad \frac{\partial z}{\partial x} = \frac{z}{yz-1}.$$

Pr: The plane $x=1$ intersects the paraboloid $z=x^2+y^2$ in a parabola. Find the slope of the tangent to the parabola at $(1,2,5)$

Soln The slope is the value of the partial derivative $\frac{\partial z}{\partial y}$ at $(1,2)$.

$$\left.\frac{\partial z}{\partial y}\right|_{(1,2)} = \left.\frac{\partial}{\partial y}(x^2+y^2)\right|_{(1,2)}$$

$$= \left.2y\right|_{(1,2)} = 4.$$

Pr If $f(x,y) = x\cos y + ye^x$, find the second order derivatives $f_{xx}, f_{xy}, f_{yx}, f_{yy}$

Soln $f_x = \cos y + ye^x$

$f_y = -x\sin y + e^x$

$$f_{xy} = \frac{\partial}{\partial y} f_x = -\sin y + e^x.$$

$$f_{yx} = \frac{\partial}{\partial x} f_y = -\sin y + e^x.$$

$$f_{xx} = ye^x. \qquad f_{yy} = -x\cos y.$$

Pr: Use the chain rule to find the derivative of $w = xy$ wrt t along the path $x = \cos t$, $y = \sin t$. What is the value of $\frac{dw}{dt}$ at $t = \pi/2$?

Soln: We apply the Chain rule to find $\frac{dw}{dt}$

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt}$$

$$= \frac{\partial}{\partial x}(xy) \cdot \frac{d}{dt}(\cos t) + \frac{\partial}{\partial y}(xy) \cdot \frac{d}{dt}(\sin t)$$

$$= y \cdot (-\sin t) + x \cdot (\cos t)$$

$$= \sin t\,(-\sin t) + \cos t\,(\cos t)$$

$$= -\sin^2 t + \cos^2 t = \cos 2t.$$

$$\left.\frac{dw}{dt}\right|_{t=\pi/2} = \cos\left(2 \cdot \frac{\pi}{2}\right) = \cos \pi = -1.$$

Pr: Find $\frac{\partial z}{\partial x}$ & $\frac{\partial z}{\partial y}$ at $(0,0,0)$ if

$$x^3 + z^2 + ye^{xz} + z\cos y = 0$$

Soln: Let $F(x,y,z) = x^3 + z^2 + ye^{xz} + z\cos y = 0$

Then

$F_x = 3x^2 + zye^{xz}$ $\quad F_y = e^{xz} - z\sin y$ and

$$F_z = 2z + xye^{xz} + \cos y$$

$\Rightarrow$

$$\frac{\partial z}{\partial x} = \frac{-F_x}{F_z} = -\frac{3x^2 + zye^{xz}}{2z + xye^{xz} + \cos y}$$

$$\frac{\partial z}{\partial y} = \frac{-F_y}{F_z} = -\frac{e^{xz} - z\sin y}{2z + xye^{xz} + \cos y}$$

$\Rightarrow$ At $(0,0,0)$ we find

$$\frac{\partial z}{\partial x} = -\frac{0}{1} = 0 \quad \& \quad \frac{\partial z}{\partial y} = -\frac{1}{1} = -1.$$

Pr: a) Find the (directional) derivative of

$$f(x,y,z) = x^3 - xy^2 - z \quad \text{at } P_0(1,1,0)$$

in the direction of $v = 2i - 3j + 6k$.

b) In what directions does f change most rapidly at P_0 and what are the rates of change in these directions?

Soln: a) $|v| = \sqrt{2^2 + (-3)^2 + 6^2} = \sqrt{49} = 7$

$$u = \frac{v}{|v|} = \frac{2}{7} i - \frac{3}{7} j + \frac{6}{7} k$$

$$f_x = (3x^2 - y^2)\Big|_{(1,1,0)} = 2$$

$$f_y = -2xy\Big|_{(1,1,0)} = -2$$

$$f_z = -1\Big|_{(1,1,0)} = -1$$

$\nabla f = \langle 2, -2, -1 \rangle$ at P_0.

$$\Rightarrow (D_u f)\Big|_{(1,1,0)} = \vec{\nabla f}\Big|_{(1,1,0)} \cdot \vec{u}$$

$$= \langle 2, -2, -1 \rangle \cdot \langle \tfrac{2}{7}, \tfrac{-3}{7}, \tfrac{6}{7} \rangle = \tfrac{4}{7}.$$

b) The fnc f increases most rapidly in the direction of $\nabla f = 2i - 2j - k$ and decreases most rapidly in the direction of $-\nabla f = \langle -2, 2, 1 \rangle$.

The rates of change in the directions are

$$|\nabla f| = \sqrt{2^2 + (-2)^2 + (-1)^2} = \sqrt{9} = 3.$$

$$-|\nabla f| = -3.$$

Pr: Find the tangent plane and normal line of the surface

$$f(x,y,z) = x^2+y^2+z-9=0$$ at

the point $P_0\ (1,2,4)$.

Soln: The tangent plane is the plane through P_0 perpendicular to the gradient of f at P_0.

$$\nabla f\big|_{P_0} = \left(2x\,i + 2y\,j + k\right)\Big|_{(1,2,4)} = \langle 2,4,1\rangle$$

$\Rightarrow$ the tangent plane is the plane

$$2(x-1) + 4(y-2) + (z-4) = 0$$

i.e $$2x+4y+z=14.$$

The normal line to the surface at P_0 is

$$x = 1+2t\ ,\quad y = 2+4t\ ,\quad z = 4+t.$$

Pr: Find the plane tangent to the surface $z = x\cos y - ye^x$ at $(0,0,0)$

Soln: let $z = f(x,y)$.

$$f_x(0,0) = \left(\cos y - ye^x\right)\Big|_{(0,0)} = 1 - 0.1 = 1$$

$$f_y(0,0) = \left(-x\sin y - e^x\right)\Big|_{(0,0)} = 0 - 1 = -1.$$

The tangent plane is

$$1(x-0) - 1(y-0) - (z-0) = 0$$

$$x - y - z = 0.$$

Pr: Find the linearization of

$$f(x,y) = x^2 - xy + \frac{1}{2}y^2 + 3$$ at the point $(3,2)$. Then use it to approximate at $(3.1, 1.9)$.

Soln: Thm: If the partial derivatives f_x & f_y exist near (a,b) and are continuous at (a,b) then f is differentiable at (a,b).

$\Rightarrow$ We first evaluate f, f_x, f_y at $(3,2)$.

$$f(3,2) = \left(x^2 - xy + \frac{1}{2}y^2 + 3\right)\Big|_{(3,2)} = 8$$

$$f_x(3,2) = \frac{\partial}{\partial x}\left(x^2 - xy + \frac{1}{2}y^2 + 3\right)\Big|_{(3,2)}$$

$$= (2x - y)\Big|_{(3,2)} = 4$$

$$f_y(3,2) = \frac{\partial}{\partial y}\left(x^2 - xy + \frac{1}{2}y^2 + 3\right)\Big|_{(3,2)}$$

$$= (-x + y)\Big|_{(3,2)} = -1.$$

Since f_x, f_y exist near $(3,2)$ and are continuous at $(3,2)$ (they are both polynomials) f is differentiable at $(3,2)$.

and the linearization

$$\mathcal{L}(x,y) = f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)$$

$$= 8 + 4(x-3) + (-1)(y-2) = 4x - y - 2.$$

$$\Rightarrow \mathcal{L}(3.1, 1.9) = 4(3.1) - (1.9) - 2$$

$$= 8.5$$

Pr: Find the local extreme values of

$$f(x,y) = 3y^2 - 2y^3 - 3x^2 + 6xy.$$

Soln: Since f is differentiable everywhere it can assume extreme values only where

$$f_x = 6y - 6x = 0 \quad (1) \qquad \text{and} \qquad f_y = 6y - 6y^2 + 6x = 0 \quad (2)$$

From (1) we get $x = y$

(2) gives $6x - 6x^2 + 6x = 0 \Rightarrow 6x(2-x) = 0$

$\Rightarrow$ Two critical points are $(0,0)$ & $(2,2)$.

Note that $f_{xx} = -6$, $f_{xy} = 6$ and

$$f_{yy} = 6 - 12y$$

$$D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = f_{xx} f_{yy} - f_{xy}^2$$
$$= (-36 + 72y) - 36$$
$$= 72(y-1).$$

At $(0,0)$, $D(0,0) = -72 \Rightarrow (0,0)$ is a saddle pt.

At $(2,2)$, $D(2,2) = 72$ & $f_{xx} = -6 \Rightarrow (2,2)$ is a local maximum

Pr: Find the greatest and smallest values that the fnc $f(x,y) = xy$ takes on the ellipse $\frac{x^2}{8} + \frac{y^2}{2} = 1$.

Soln: We want to find the extreme values of $f(x,y) = xy$ subject to the constraint

$$g(x,y) = \frac{x^2}{8} + \frac{y^2}{2} - 1 = 0$$

$\Rightarrow$ To do so we first find the values of x, y and λ s.t

$$\nabla f = \lambda \nabla g \quad \text{and} \quad g(x,y) = 0.$$

$$\Rightarrow \begin{cases} y = \frac{\lambda}{4} x & (1) \\ x = \lambda y & (2) \\ \frac{x^2}{8} - \frac{y^2}{2} - 1 = 0 \end{cases}$$

$\Rightarrow$ (1) & (2) gives $y = \frac{\lambda}{4}(\lambda y) = \frac{\lambda^2}{4} y$

so that $y = 0$ or $\lambda = \pm 2$

We now consider these two cases:

<u>Case 1</u> If $y=0$ then $x=y=0$ but $(0,0)$ is not on the ellipse $\Rightarrow y \neq 0$

<u>Case 2</u> If $y \neq 0$ then $\lambda = \mp 2$ and $x = \mp 2y$. Substituting this in the eqn $g(x,y)=0$ gives

$$\frac{(\mp 2y)^2}{8} + \frac{y^2}{2} = 1 \;, \quad 4y^2 + 4y^2 = 8$$

$\Rightarrow y = \mp 1.$

The fnc $f(x,y) = xy$ therefore takes on its extreme values on the ellipse at the four points $(\mp 2, 1), (\mp 2, -1)$.

The extreme values are $xy = 2$ and $xy = -2$.

Pr: Find the volume of the region bounded by the elliptical paraboloid $z = 10 + x^2 + 3y^2$ and below by the rectangle $R: 0 \le x \le 1$ & $0 \le y \le 2$

Soln:

$$V = \iint_R (10 + x^2 + 3y^2)\, dA$$

$$= \int_0^1 \int_0^2 (10 + x^2 + 3y^2)\, dy\, dx$$

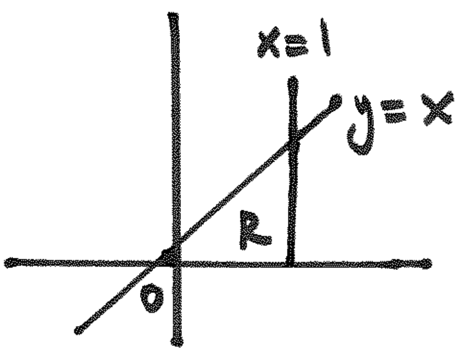
$$= \int_0^1 \left[10y + x^2 y + y^3 \right]_{y=0}^{y=2} dx$$

$$= \int_0^1 (20 + 2x^2 + 8)\, dx$$

$$= 20x + \frac{2}{3}x^3 + 8x \Big|_0^1 = \frac{86}{3}.$$

Pr: Calculate $\iint_R \frac{\sin x}{x}\, dA$

where R is the triangle in the xy plane bounded by the x-axis, the line $y=x$ & the line $x=1$.



$$\int_0^1 \int_0^x \frac{\sin x}{x}\, dy\, dx = \int_0^1 y\frac{\sin x}{x}\Big|_{y=0}^{y=x} dx$$

$$= \int_0^1 \sin x\, dx = -\cos(1)+1.$$

Note that $\int_0^1 \int_y^1 \frac{\sin x}{x}\, dx\, dy$ is problematic

there is no simple antiderivative.

Pr: Find the volume of the solid region bounded above by the paraboloid $z = 9 - x^2 - y^2$ and below by the unit circle in the xy-plane.

$$V = \iint_R (9 - x^2 - y^2)\, dA \quad \text{where}$$

R is the unit circle $x^2 + y^2 = 1$.

Switch to polar coords:

$$\int_0^{2\pi} \int_0^1 (9 - r^2)\, r \, dr\, d\theta$$

$$= \int_0^{2\pi} \int_0^1 (9r - r^3)\, dr\, d\theta$$

$$= \int_0^{2\pi} \left[\frac{9}{2} r^2 - \frac{1}{4} r^4 \right]_{r=0}^{r=1} d\theta$$

$$= \frac{17}{4} \int_0^{2\pi} d\theta = \frac{17\pi}{2}.$$