Midterm 2 Soln

1. (9 points) For parts (a)-(c) of this problem, no work is required and there is no partial credit. **Put your answer in the answer box.** Let $f(x, y, z) = x\sin(3yz)$. Compute the following partial derivatives.

(a) $f_z =$

$$f_z = 3xy\cos(3yz)$$

(b) $f_{zx} =$

$$f_{zx} = 3y\cos(3yz)$$

(c) $f_{zxy} =$

$$f_{zxy} = 3\cos(3yz) - 9yz\sin(3yz)$$

2. (10 points) Find an equation for the tangent plane to the ellipsoid

$$x^2 + y^2 + 2z^2 = 10$$

at the point $(1, 1, 2)$.

$\rightarrow$ ∇f is normal to $f(x,y,z) = 10$

$f = x^2 + y^2 + 2z^2 \Rightarrow \nabla f = \langle 2x, 2y, 4z \rangle$

$\vec{n} = \nabla f(1,1,2) = \langle 2, 2, 8 \rangle.$

$$\Rightarrow \boxed{2(x-1) + 2(y-1) + 8(z-2) = 0}$$

3. (14 points) Let $f(x, y) = x^2 + 3y^2$.

(a) Find the directional derivative of f in the direction of the vector $\mathbf{v} = \langle 1, -1 \rangle$ at the point $(2, 1)$.

$\rightarrow$ For a unit vec. $\vec{u}$, $D_{\vec{u}} f = \nabla f \cdot \vec{u}$.

(a) Find the directional derivative of f in the direction of the vector $\mathbf{v} = \langle 1, -1 \rangle$ at the point $(2, 1)$.

$\rightarrow$ For a unit vec. $\vec{u}$, $D_{\vec{u}} f = \nabla f \cdot \vec{u}$.

The unit vector in the direction of $\vec{v}$ is

$\vec{u} = \frac{1}{\sqrt{2}} \langle 1, -1 \rangle$. Then, $\nabla f = \langle 2x, 6y \rangle$

$\Rightarrow \nabla f(2,1) = \langle 4, 6 \rangle$.

Hence, $D_{\vec{u}} f = \langle 4, 6 \rangle \cdot \frac{1}{\sqrt{2}} \langle 1, -1 \rangle$

$$= \frac{4-6}{\sqrt{2}} = \boxed{-\sqrt{2}}$$

(b) Find the unit vector in the direction for which $f(x, y)$ is increasing fastest at the point $(2, 1)$.

$\rightarrow$ ∇f points in the direction of fastest increase. So,

$$\vec{w} = \frac{\nabla f(2,1)}{|\nabla f(2,1)|} = \frac{\langle 4,6 \rangle}{\sqrt{4^2+6^2}} = \frac{\langle 4,6 \rangle}{\sqrt{52}} = \boxed{\frac{\langle 2,3 \rangle}{\sqrt{13}}}$$

4. (20 points) Complete each part below, showing all work.

(a) Evaluate the double integral $\iint_D e^x \cos y \, dA$ where D is the rectangular region

$$D = \{(x, y) \in \mathbb{R}^2 \,|\, 1 \le x \le 2, 0 \le y \le \pi/3\}.$$

$$\rightarrow \iint_D e^x \cos y \, dA = \int_{x=1}^{2} \int_{y=0}^{\pi/3} e^x \cos y \, dy \, dx$$

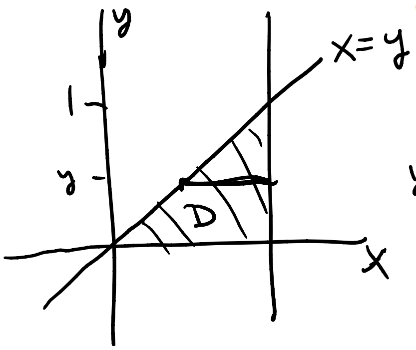
$$= \int_1^2 e^x dx \int_0^{\pi/3} \cos y \, dy$$

$$= e^x \Big|^2 \sin y \Big|^{\pi/3}$$

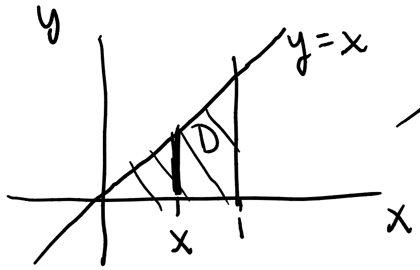
$$= e^x \Big|_1^2 \; \sin y \Big|_0^{\pi/3}$$

$$= \boxed{\frac{(e^2 - e)\sqrt{3}}{2}}$$

(b) Evaluate the iterated integral $\int_0^1 \int_y^1 e^{x^2}\, dx\, dy$.



$$\int_{y=0}^{1} \int_{x=y}^{1} e^{x^2}\, dx\, dy = \iint_D e^{x^2}\, dA$$



$$= \int_{x=0}^{1} \int_{y=0}^{x} e^{x^2}\, dy\, dx$$

$$= \int_{x=0}^{1} x e^{x^2}\, dx$$

$$= \int_{u=0}^{1} \frac{1}{2} e^u\, du$$

$u = x^2$

$\frac{1}{2} du = x\, dx$

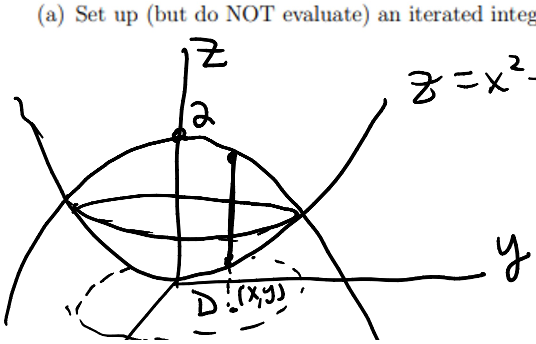
$\underline{x=0}$: $u = 0^2 = 0$

$\underline{x=1}$: $u = 1^2 = 1$

$$= \boxed{\frac{e-1}{2}}$$

5. (15 points) Let S be the solid bounded by the surfaces $z = x^2 + y^2$ and $z = 2 - x^2 - y^2$.

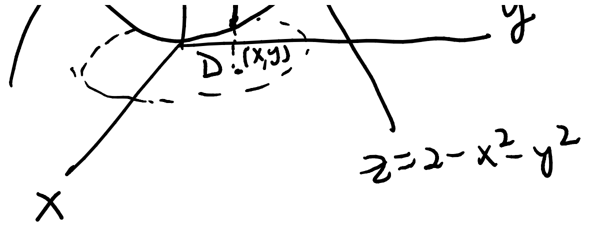
(a) Set up (but do NOT evaluate) an iterated integral in Cartesian coordinates to find the volume of S.



Intersection of surfaces:

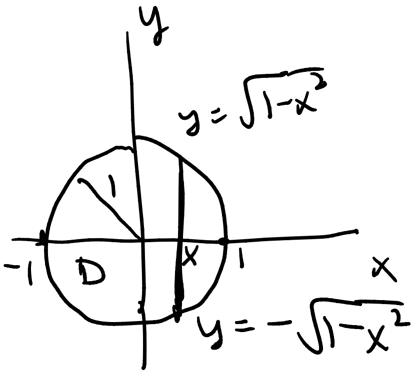
$x^2 + y^2 = 2 - (x^2 + y^2)$

$\Rightarrow x^2 + y^2 = 1$



$$\Rightarrow x^2+y^2=1$$

Let $D=\{(x,y)\mid x^2+y^2\le 1\}$.



$$\Rightarrow \mathrm{vol}(S)=\iint_D\left[2-x^2-y^2-(x^2+y^2)\right]dA$$

$$=\boxed{\int_{x=-1}^{1}\int_{y=-\sqrt{1-x^2}}^{\sqrt{1-x^2}}2(1-x^2-y^2)\,dy\,dx}$$

(b) Set up (but do NOT evaluate) an iterated integral in polar coordinates to find the volume of S.

In polar, $D=\{(r,\theta)\mid 0\le r\le 1,\ 0\le\theta<2\pi\}$

$$\to \mathrm{vol}(S)=\iint_D 2(1-x^2-y^2)\,dA$$

$$=\boxed{\int_{\theta=0}^{2\pi}\int_{r=0}^{1}2(1-r^2)\,r\,dr\,d\theta}$$

(c) Evaluate either integral in parts (a) or (b) to find the volume of S.

$$\to \mathrm{vol}(S)=\int_{\theta=0}^{2\pi}\int_{r=0}^{1}2(1-r^2)\,r\,dr\,d\theta$$

$$=2\int_0^{2\pi}d\theta\int_0^1(r-r^3)\,dr$$

$$=4\pi\left(\tfrac{1}{2}r^2-\tfrac{1}{4}r^4\right)\Big|^1$$

$$= 4\pi\left(\frac{1}{2}r^2 - \frac{1}{4}r^4\right)\Big|_{r=0}^{1}$$

$$= 4\pi\left(\frac{1}{2} - \frac{1}{4}\right) = \boxed{\pi}$$

6. (16 points) Find all critical points of

$$f(x, y) = x^3 - 3x + y^4 - 2y^2$$

and classify each as a local maximum, minimum, or saddle point.

$\rightarrow$ $f_x = 3x^2 - 3 = 3(x^2-1) \Rightarrow f_{xx} = 6x,\ f_{xy} = 0$

$f_x = 0 \Rightarrow x = \pm 1$

$f_y = 4y^3 - 4y = 4y(y^2-1) \Rightarrow f_{yy} = 12y^2 - 4$

$f_y = 0 \Rightarrow y = -1, 0, 1$

Critical pts: $(1,1), (1,-1), (1,0)$
$(-1,1), (-1,-1), (-1,0)$

$D(x,y) = f_{xx}f_{yy} - f_{xy}^2$
$= 6x(12y^2-4) - 0^2$
$= 24x(3y^2-1)$

- $D(1,1) > 0$ and $f_{xx}(1,1) > 0 \Rightarrow (1,1)$ local min.
- $D(1,-1) > 0$ and $f_{xx}(1,-1) > 0 \Rightarrow (1,-1)$ local min.
- $D(1,0) < 0 \Rightarrow (1,0)$ a saddle.
- $D(-1,1) < 0 \Rightarrow (-1,1)$ a saddle.
- $D(-1,-1) < 0 \Rightarrow (-1,1)$ a saddle.
- $D(-1,0) > 0$ and $f_{xx}(-1,0) < 0 \Rightarrow (-1,0)$ local max.

7. (16 points) Use the method of Lagrange multipliers to find all points on the cone $y^2 = x^2 + z^2$ that are closest to the point $(2, 0, 0)$. Note: you will lose significant points if you do NOT use Lagrange multipliers (even if you obtain the correct answer).

$\rightarrow$ Distance from (x,y,z) to $(2,0,0) = \sqrt{(x-2)^2 + y^2 + z^2}$.

We minimize $f(x,y,z) = (x-2)^2 + y^2 + z^2$ subject to the constraint $x^2 - y^2 + z^2 = 0$.

Let $g(x,y,z) = x^2 - y^2 + z^2$. Then:

$$\begin{cases} \nabla f = \lambda \nabla g \\ g = 0 \end{cases} \Rightarrow \begin{cases} 2(x-2) = \lambda(2x) \\ 2y = -\lambda(2y) \\ 2z = \lambda(2z) \\ x^2 - y^2 + z^2 = 0 \end{cases}$$

$$\Rightarrow \begin{cases} x - 2 = \lambda x \\ y(1+\lambda) = 0 \\ z(1-\lambda) = 0 \\ x^2 + z^2 = y^2 \end{cases}$$

$\longrightarrow$ Either $y = 0$ or $\lambda = -1$.

If $y = 0$, then $x^2 + z^2 = 0$ $\Rightarrow x = z = 0$.

But then $x - 2 = \lambda x$ implies $-2 = 0$ which is a contradiction.

So $y \neq 0$ and $\lambda = -1$.

$\lambda = -1 \Rightarrow x - 2 = \lambda x = -x \Rightarrow 2x = 2 \Rightarrow \boxed{x = 1}$

Also, $z(1-\lambda) = 0 \Rightarrow 2z = 0 \Rightarrow \boxed{z = 0}$

Then, $x^2 + z^2 = y^2 \Rightarrow y^2 = 1 \Rightarrow \boxed{y = \pm 1}$

<u>Sol'ns</u>: $\lambda = -1$ w/ $(x,y,z) = (1,1,0)$ and $(x,y,z) = (1,-1,0)$.

$f(1,1,0) = (1-2)^2 + 1^2 + 0^2 = 2$

$f(1,-1,0) = (1-2)^2 + (-1)^2 + 0^2 = 2$.

Since the minimum distance must arise from a sol'n to Lagr. equations we see that $(1, \pm 1, 0)$ are minima and a sol'n to Lagr. equations we see that both $(1,1,0)$ and $(1,-1,0)$ are minima and the min. distance $= \sqrt{f} = \sqrt{2}$.