1. (9 points) For parts (a)-(c) of this problem, no work is required and there is no partial credit. **Put your answer in the answer box.** Let $f(x, y, z) = x\sin(3yz)$. Compute the following partial derivatives.

(a) $f_z =$

(b) $f_{zx} =$

(c) $f_{zxy} =$

2. (10 points) Find an equation for the tangent plane to the ellipsoid

$$x^2 + y^2 + 2z^2 = 10$$

at the point $(1, 1, 2)$.

3. (14 points) Let $f(x, y) = x^2 + 3y^2$.

(a) Find the directional derivative of f in the direction of the vector $\mathbf{v} = \langle 1, -1 \rangle$ at the point $(2, 1)$.

(b) Find the unit vector in the direction for which $f(x, y)$ is increasing fastest at the point $(2, 1)$.

4. (20 points) Complete each part below, showing all work.

(a) Evaluate the double integral $\iint_D e^x \cos y \, dA$ where D is the rectangular region

$$D = \{ (x, y) \in \mathbb{R}^2 \,|\, 1 \leq x \leq 2, 0 \leq y \leq \pi/3 \}.$$

$$\rightarrow \iint_D e^x \cos y \, dA = \int_{x=1}^{2} \int_{y=0}^{\pi/3} e^x \cos y \, dy \, dx$$

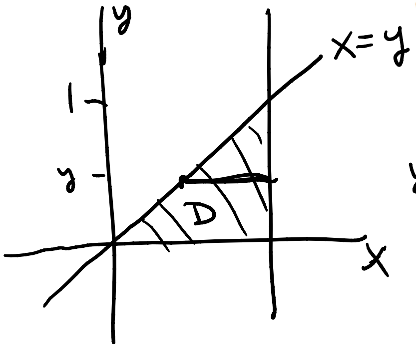
$$= \int_1^2 e^x \, dx \int_0^{\pi/3} \cos y \, dy$$

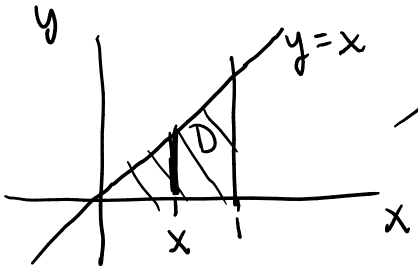
$$= e^x \Big|^{2} \sin y \Big|^{\pi/3}$$

$$= e^x\Big|_1^2 \sin y\Big|_0^{\pi/3}$$

$$= \boxed{\frac{(e^2-e)\sqrt{3}}{2}}$$

(b) Evaluate the iterated integral $\int_0^1 \int_y^1 e^{x^2}\, dx\, dy$.





$$\int_{y=0}^{1}\int_{x=y}^{1} e^{x^2}\,dx\,dy = \iint_D e^{x^2}\,dA$$

$$= \int_{x=0}^{1}\int_{y=0}^{x} e^{x^2}\,dy\,dx$$

$$= \int_{x=0}^{1} x e^{x^2}\,dx$$

$$= \int_{u=0}^{1} \tfrac{1}{2} e^{u}\,du$$

$$= \boxed{\frac{e-1}{2}}$$

$u = x^2$

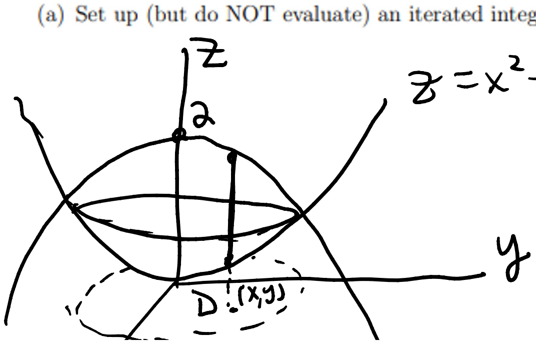
$\frac{1}{2}du = x\,dx$

$\underline{x=0}$: $u = 0^2 = 0$

$\underline{x=1}$: $u = 1^2 = 1$

5. (15 points) Let S be the solid bounded by the surfaces $z = x^2 + y^2$ and $z = 2 - x^2 - y^2$.

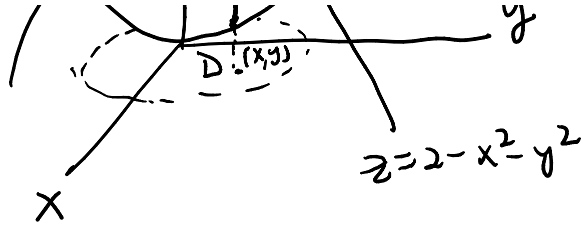
(a) Set up (but do NOT evaluate) an iterated integral in Cartesian coordinates to find the volume of S.



Intersection of surfaces:

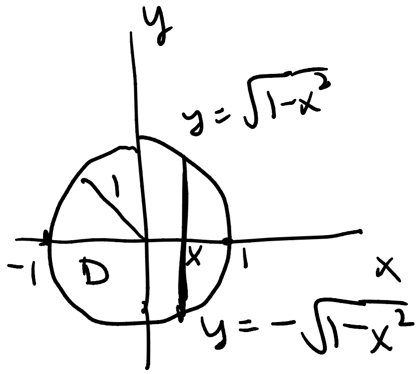
$$x^2+y^2 = 2-(x^2+y^2)$$

$$\Rightarrow x^2+y^2 = 1$$



$\Rightarrow x^2+y^2=1$

Let $D=\{(x,y) \mid x^2+y^2 \le 1\}$.



$$\Rightarrow \text{vol}(S) = \iint_D \left[2-x^2-y^2-(x^2+y^2)\right] dA$$

$$= \boxed{\int_{x=-1}^{1} \int_{y=-\sqrt{1-x^2}}^{\sqrt{1-x^2}} 2(1-x^2-y^2)\, dy\, dx}$$

(b) Set up (but do NOT evaluate) an iterated integral in polar coordinates to find the volume of S.

In polar, $D=\{(r,\theta) \mid 0 \le r \le 1,\ 0 \le \theta < 2\pi\}$

$$\rightarrow \text{vol}(S) = \iint_D 2(1-x^2-y^2)\, dA$$

$$= \boxed{\int_{\theta=0}^{2\pi} \int_{r=0}^{1} 2(1-r^2)\, r\, dr\, d\theta}$$

(c) Evaluate either integral in parts (a) or (b) to find the volume of S.

$$\rightarrow \text{vol}(S) = \int_{\theta=0}^{2\pi} \int_{r=0}^{1} 2(1-r^2)\, r\, dr\, d\theta$$

$$= 2\int_0^{2\pi} d\theta \int_0^1 (r-r^3)\, dr$$

$$= 4\pi\left(\tfrac{1}{2}r^2 - \tfrac{1}{4}r^4\right)\Big|^1$$

$$= 4\pi\left(\frac{1}{2}r^2 - \frac{1}{4}r^4\right)\Big|_{r=0}^{1}$$

$$= 4\pi\left(\frac{1}{2} - \frac{1}{4}\right) = \boxed{\pi}$$

6. (16 points) Find all critical points of

$$f(x, y) = x^3 - 3x + y^4 - 2y^2$$

and classify each as a local maximum, minimum, or saddle point.

7. (16 points) Use the method of Lagrange multipliers to find all points on the cone $y^2 = x^2 + z^2$ that are closest to the point $(2, 0, 0)$. Note: you will lose significant points if you do NOT use Lagrange multipliers (even if you obtain the correct answer).

$\rightarrow$ Distance from (x,y,z) to $(2,0,0) = \sqrt{(x-2)^2 + y^2 + z^2}$.

We minimize $f(x,y,z) = (x-2)^2 + y^2 + z^2$ subject to the constraint $x^2 - y^2 + z^2 = 0$.

Let $g(x,y,z) = x^2 - y^2 + z^2$. Then:

$$\begin{cases} \nabla f = \lambda \nabla g \\ g = 0 \end{cases} \Rightarrow \begin{cases} 2(x-2) = \lambda(2x) \\ 2y = -\lambda(2y) \\ 2z = \lambda(2z) \\ x^2 - y^2 + z^2 = 0 \end{cases}$$

$$\Rightarrow \begin{cases} x - 2 = \lambda x \\ y(1+\lambda) = 0 \\ z(1-\lambda) = 0 \\ x^2 + z^2 = y^2 \end{cases}$$

$\longrightarrow$ Either $y = 0$ or $\lambda = -1$.

If $y = 0$, then $x^2 + z^2 = 0$ $\Rightarrow x = z = 0$.

But then $x - 2 = \lambda x$ implies $-2 = 0$ which is a contradiction.

So $y \neq 0$ and $\lambda = -1$.

$\lambda = -1 \Rightarrow x - 2 = \lambda x = -x \Rightarrow 2x = 2 \Rightarrow \boxed{x = 1}$

Also, $z(1-\lambda) = 0 \Rightarrow 2z = 0 \Rightarrow \boxed{z = 0}$

Then, $x^2 + z^2 = y^2 \Rightarrow y^2 = 1 \Rightarrow \boxed{y = \pm 1}$

Sol'ns: $\lambda = -1$ w/ $(x,y,z) = (1,1,0)$ and $(x,y,z) = (1,-1,0)$.

$f(1,1,0) = (1-2)^2 + 1^2 + 0^2 = 2$

$f(1,-1,0) = (1-2)^2 + (-1)^2 + 0^2 = 2$.

Since the minimum distance must arise from a sol'n to Lagr. equations we see that $(1, \pm 1, 0)$ are minima and a sol'n to Lagr. equations we see that both $(1,1,0)$ and $(1,-1,0)$ are minima and the min. distance $= \sqrt{f} = \sqrt{2}$.