Math 164: Multidimensional Calculus

Midterm II

November 17, 2016

NAME (please print legibly): SOLUTIONS

Your University ID Number: ______________________

Indicate your instructor with a check in the appropriate box:

Kleene	TR 12:30-1:45pm	
Salur	MW 3:25-4:40pm	
Gafni	TR 3:25-4:40pm	
Lee	MWF 09:00-09:50am	

- You have 75 minutes to work on this exam.
- You are responsible for checking that this exam has all 10 pages.
- No calculators, phones, electronic devices, books, notes are allowed during the exam.
- Show all work and justify all answers.
- Please sign the pledge below.

Pledge of Honesty

I affirm that I will not give or receive any unauthorized help on this exam, and that all work will be my own.

Signature: ______________________

QUESTION	VALUE	SCORE
1	10	
2	15	
3	10	
4	15	
5	25	
6	15	
7	10	
TOTAL	100	

1. (10 points) Consider the function

$$f(x,y,z) = \frac{2x^2}{x+z} + x^2y^2e^z - xy\sin(z) + \frac{e^{\sqrt{x+y}}}{\cos(xy)}.$$

← don't want to differentiate that! Use Clairaut!

Evaluate the third order partial derivative $f_{xyz}(1,1,0)$.

$$f_z = -2x^2(x+z)^{-2} + x^2y^2e^z - xy\cos(z) + 0$$

$$\Rightarrow (f_z)_y = 0 + 2x^2ye^z - x\cos(z)$$

$$\Rightarrow f_{zyx} = 4xye^z - \cos(z)$$

$$\Rightarrow f_{zyx}(1,1,0) = 4\cdot 1\cdot 1e^0 - \cos(0)$$
$$= 4-1$$
$$= \boxed{3}$$

We can see all mixed partials will be continuous at $(1,1,0)$ since all functions involved are continuous everywhere except where their denominators $=0$, which could at worst be where $x+z=0$ or $\cos(xy)=0$, neither of which is true for $(1,1,0)$. Thus, by Clairaut, all mixed partials are equal at $(1,1,0)$, so $f_{xyz}(1,1,0) = f_{zyx}(1,1,0) = \boxed{3}$

2. (15 points) Let $f(x, y)$ be given by

$$f(x, y) = 4 - \ln(xy - 9) + x^2 cos(\pi y).$$

a) Explain why $f(x, y)$ is differentiable at $(5, 2)$.

$$f_x = -\frac{y}{xy-9} + 2x\cos(\pi y)$$

$$f_y = -\frac{x}{xy-9} - \pi x^2 \sin(\pi y)$$

If the partial derivs exist near (a,b) and are continuous ~~at~~ at (a,b) then f is differentiable at (a,b). In this case, f_x and f_y exist and are continuous away from $xy=9$, so f is differentiable at (5,2) since $5 \cdot 2 \neq 9$.

b) Find an equation for the linearization of f at the point $(5, 2)$, and use it to estimate the value of f at the point $(5.05, 1.98)$.

$$L(x,y) = f_x(5,2)(x-5) + f_y(5,2)(y-2) + f(5,2)$$

$$= \left[-\frac{2}{10-9} + 10(-1)\right](x-5) + \frac{-5}{10-9}(y-2) + (4 - \ln(1) + 25(-1))$$

$$= -12(x-5) - 5(y-2) + (-21)$$

Near (5,2), $f(x,y) \approx L(x,y)$ so

$$f(5.05, 1.98) \approx L(5.05, 1.98) = -12(.05) - 5(-.02) - 21$$

$$= -.60 - .10 - 21$$

$$= \boxed{-21.7}$$

$$f(5.05, 1.98) \approx -21.7$$

3. (10 points) Write an equation for the tangent plane to the surface $x^2 + 2y^2 + 3z^2 = 6$ at the point $(1, 1, 1)$.

$f(x,y,z) = x^2 + 2y^2 + 3z^2$

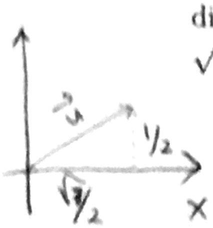
$\nabla f = \langle f_x, f_y, f_z \rangle = \langle 2x, 4y, 6z \rangle$

$\nabla f(1,1,1) = \langle 2,4,6 \rangle = \vec{n}$ = normal vector to tangent plane at $(1,1,1)$

$\vec{n} \cdot \langle x-1, y-1, z-1 \rangle = 0$

$2(x-1) + 4(y-1) + 6(z-1) = 0$

4. (15 points) Find the directional derivative of $f(x, y) = 2x^2y^3 + 6xy$ at (1,1) in the direction of a unit vector whose angle with the positive x-axis is $\pi/6$. (Recall that $\cos(\pi/6) = \sqrt{3}/2$)



$\vec{u} = \langle \frac{\sqrt{3}}{2}, \frac{1}{2} \rangle$

$$D_{\vec{u}}(1,1) = \nabla f(1,1) \cdot \vec{u}$$
$$= f_x(1,1)\cos\theta + f_y(1,1)\sin\theta$$
$$= 10 \cdot \frac{\sqrt{3}}{2} + 12 \cdot \frac{1}{2}$$
$$= \boxed{5\sqrt{3} + 6}$$

$f_x = 4xy^3 + 6y \longrightarrow f_x(1,1) = 4+6 = 10$

$f_y = 6x^2y^2 + 6x \longrightarrow f_y(1,1) = 6+6 = 12$

$\Rightarrow \nabla f = \langle 10, 12 \rangle$ at $(1,1)$

5. (25 points) **a)** Consider the function $f(x,y) = x^2y - 2xy + 2y^2 - 15y$. Find all the critical points of f and classify them as relative maxima, relative minima, or saddle points.

$$f_x = 2xy - 2y + 0 - 0 = 2xy - 2y = 2(x-1)y$$

$\Rightarrow f_x = 0$ when $x = 1$ or $y = 0$

$$f_y = x^2 - 2x + 4y - 15$$

$\Rightarrow f_y = 0$ when $x^2 - 2x + 4y - 15 = 0$

If $\underline{x=1}$, $f_y = 0 \Rightarrow 4y - 16 = 0 \Rightarrow y = 4$ $\boxed{(1,4)}$

If $\underline{y=0}$, $f_y = 0 \Rightarrow x^2 - 2x - 15 = 0 \Rightarrow (x+3)(x-5) = 0$

$\Rightarrow x = -3$ or $x = 5$

$\boxed{(-3,0)}$ and $\boxed{(5,0)}$

f_x and f_y are defined everywhere so the only critical pts. are where $f_x = 0 = f_y$: $(1,4)$, $(-3,0)$, and $(5,0)$

$f_{xx} = 2y$; $f_{yy} = 4$; $f_{xy} = f_{yx} = 2x - 2$ $\Rightarrow D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = f_{xx}f_{yy} - f_{xy}^2$

$= 8y - (2x-2)^2$

$= 8y - (4x^2 - 8x + 4)$

At $(1,4)$, $D = 32 - 0 = 32 > 0$ and $f_{xx} = 8 > 0$

so $\boxed{(1,4) \text{ is a rel min.}}$

At $(-3,0)$, $D = 0 - (2(-3)-2)^2 < 0$, so $\boxed{(-3,0) \text{ is a saddle pt.}}$

At $(5,0)$, $D = 0 - (2(5)-2)^2 < 0$, so $\boxed{(5,0) \text{ is a saddle pt.}}$

b) Find the extreme values of the function $f(x,y) = 3x^2 + 2y^2 - 4y$ subject to the constraint $x^2 + y^2 \leq 9$. Let $D = \{(x,y) \mid x^2+y^2 \leq 9\}$. D is closed and bdd so extrema are attained.

On the boundary, use Lagrange multipliers ($g(x,y) = x^2+y^2 = 9$):

$$\left.\begin{array}{l}\nabla f = \langle 6x, 4y-4\rangle \\ \nabla g = \langle 2x, 2y\rangle\end{array}\right\} \Rightarrow \nabla f = \lambda \nabla g \text{ gives } \begin{array}{l} 6x = \lambda 2x \\ 4y-4 = \lambda 2y\end{array}$$

① $6x = \lambda 2x \Rightarrow 2x(3-\lambda) = 0 \Rightarrow x = 0$ or $\lambda = 3$

② $4y - 4 = \lambda 2y \Rightarrow 4y - 2\lambda y - 4 = 0$

③ $x^2 + y^2 = 9$

① and ② $\Rightarrow$ if $\underline{\lambda = 3}$, $4y - 6y - 4 = 0 \Rightarrow -2y = 4 \Rightarrow y = -2 \overset{\text{by ③}}{\Rightarrow} x = \pm\sqrt{5} \Rightarrow$

$\boxed{(\sqrt{5}, -2)} \rightarrow f = 15 - 4 + 8 = \boxed{19}$

$\boxed{(-\sqrt{5}, -2)} \rightarrow f = \boxed{19}$

① and ③ $\Rightarrow$ if $\underline{x = 0}$, $y^2 = 9 \Rightarrow y = \pm 3 \Rightarrow$

$\boxed{(0,3)} \rightarrow f = 0 + 18 - 12 = \boxed{6}$

$\boxed{(0,-3)} \rightarrow f = 0 + 18 + 12 = \boxed{30}$

On the interior, check for critical points: $f_x = 0$ when $x = 0$

$f_y = 0$ when $4y = 4$, or $y = 1$

$\boxed{(0,1)} \rightarrow f = 0 + 2 - 4 = \boxed{-2}$

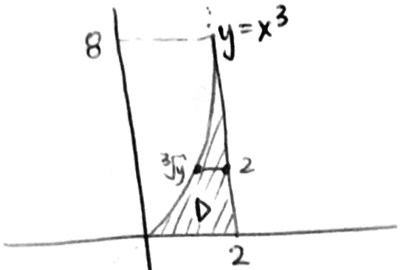
So, we must compare the values $-2, 6, 19$, and 30

$\boxed{\begin{array}{l}\text{The abs. max. of } f \text{ on } D \text{ is } 30, \text{ attained at } (0,-3). \\ \text{The abs. min. of } f \text{ on } D \text{ is } -2, \text{ attained at } (0,1).\end{array}}$

6. (15 points) Evaluate the integral by reversing the order of integration.

$x = \sqrt[3]{y} \Rightarrow y = x^3$

$$I = \int_0^8 \int_{\sqrt[3]{y}}^2 e^{x^4}\,dx\,dy = \int_0^2 \int_0^{x^3} e^{x^4}\,dy\,dx$$



$D = \{(x,y) \mid 0 \le y \le 8,\ \sqrt[3]{y} \le x \le 2\}$

$D = \{(x,y) \mid 0 \le 2 \le x,\ 0 \le y \le x^3\}$

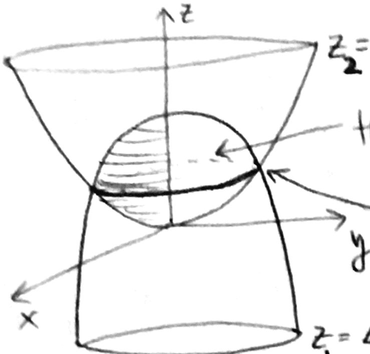
$$I = \int_0^2 \left[y e^{x^4}\right]_0^{x^3} dx = \int_0^2 x^3 e^{x^4}\,dx = \left[\frac{1}{4} e^{x^4}\right]_0^2 = \frac{1}{4}\left(e^{16} - e^0\right)$$

$u = x^4$

$\frac{1}{4}du = x^3 dx$

$$= \boxed{\frac{1}{4}\left(e^{16} - 1\right)}$$

7. (10 points) Use polar coordinates to find the volume of the solid bounded by the paraboloids $z = 3x^2 + 3y^2$ and $z = 4 - x^2 - y^2$.



$z_2 = 3x^2 + 3y^2 = 3r^2$

this is the volume we need to find.

$z_1 = 4 - (x^2+y^2) = 4 - r$

the curve of intersection lies above $3x^2 + 3y^2 = 4 - x^2 - y^2$

or $4x^2 + 4y^2 = 4$

or $x^2 + y^2 = 1$

or $r = 1$

so $D = \{(x,y) \mid x^2+y^2 \le 1\}$

or $D = \{(r,\theta) \mid 0 \le r^2 \le 1,\ 0 \le \theta \le 2\pi\}$
$= \{(r,\theta) \mid 0 \le r \le 1,\ 0 \le \theta \le 2\pi\}$

$$\text{Volume} = \iint_D (z_1 - z_2)\, dA$$

$$= \int_0^{2\pi}\int_0^1 \left((4-r) - 3r^2\right) r\, dr\, d\theta$$

$$= \int_0^{2\pi}\int_0^1 (4r - r^2 - 3r^3)\, dr\, d\theta = \int_0^{2\pi} \left[2r^2 - \frac{1}{3}r^3 - \frac{3}{4}r^4\right]_0^1 d\theta$$

$$= \left(2 - \frac{1}{3} - \frac{3}{4}\right) 2\pi = \boxed{\frac{26}{12}\pi}$$