Useful Formulas

(1) Scalar projection of $\mathbf{v}$ onto $\mathbf{u}$:

$$\text{comp}_{\mathbf{u}}(\mathbf{v}) = \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}|}$$

Vector projection of $\mathbf{v}$ onto $\mathbf{u}$:

$$\text{proj}_{\mathbf{u}}(\mathbf{v}) = \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}|^2}\mathbf{u}$$

(2) Vector equation of a line:

$$\mathbf{r}(t) = \mathbf{r}_0 + t\mathbf{v}$$

(3) Vector equation of a plane:

$$\mathbf{n} \cdot (\mathbf{r} - \mathbf{r}_0) = 0$$

(4) Distance between (x_1, y_1, z_1) and (x_2, y_2, z_2):

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

(5) The distance from a point (x_1, y_1, z_1) to the plane $ax + by + cz + d = 0$ is

$$D = \frac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2 + b^2 + c^2}}$$

(6) Arc Length

$$L = \int_a^b |\mathbf{r}'(t)| \, dt$$

(7) Implicit Differentiation for $z = f(x, y)$:

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} \qquad \frac{\partial z}{\partial y} = -\frac{F_y}{F_z}$$

(8) Gradient of $f(x, y, z)$:

$$\nabla f(x, y, z) = \langle f_x, f_y, f_z \rangle$$

(9) The tangent plane to the surface $z = f(x, y)$ at the point (x_0, y_0, z_0) is

$$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

(10) The directional derivative of f in the direction of $\mathbf{u}$:

$$D_{\mathbf{u}} f = \nabla f \cdot \mathbf{u}$$

(11) Line integral of vector field $\mathbf{F}$ over curve C:

$$\int_C \mathbf{F} \cdot \, d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) \, dt$$

(12) Fundamental theorem of line integrals:

$$\int_C \nabla f \cdot \, d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a))$$

(13) Green's Theorem: for $\mathbf{F} = P(x, y)\mathbf{i} + Q(x, y)\mathbf{j}$,

$$\iint_D \begin{vmatrix} \partial_x & \partial_y \\ P & Q \end{vmatrix} = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA = \oint \mathbf{F} \cdot d\mathbf{r}$$

(14) Divergence and Curl: for $\mathbf{F}(x, y, z) = P(x, y, z)\mathbf{i} + Q(x, y, z)\mathbf{j} + R(x, y, z)\mathbf{k}$,

$$\text{div}\, \mathbf{F} = \nabla \cdot \mathbf{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} \qquad \text{curl}\, \mathbf{F} = \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$$

(15) Parameterization of sphere of radius a: $\mathbf{r}(\phi, \theta) = \langle a \sin(\phi) \cos(\theta), a \sin(\phi) \sin(\theta), a \cos(\phi) \rangle$

(16) Scalar surface area element on a surface S given by $\mathbf{r}(u, v) = x(u, v)\mathbf{i} + y(u, v)\mathbf{j} + z(u, v)\mathbf{k}$

$$dS = |\mathbf{r}_u \times \mathbf{r}_v| \, dudv$$

- Special case S = sphere of radius a:

$$|\mathbf{r}_\phi \times \mathbf{r}_\theta| = a^2 \sin(\phi)$$

- Special case S = graph of $z = g(x, y)$

$$|\mathbf{r}_x \times \mathbf{r}_y| = \sqrt{g_x^2 + g_y^2 + 1}$$

(17) Corresponding vector surface area element

$$\mathbf{n} dS = \pm(\mathbf{r}_u \times \mathbf{r}_v)\, dudv$$

- Special case S = sphere of radius a:

$$\mathbf{r}_\phi \times \mathbf{r}_\theta = a^2 \langle \sin^2(\phi)\cos(\theta), \sin^2(\phi)\sin(\theta), \sin(\phi)\cos(\phi)\rangle$$

- Special case S = graph of $z = g(x, y)$

$$\mathbf{r}_x \times \mathbf{r}_y = \langle -g_x, -g_y, 1\rangle$$

(18) Stokes Theorem:

$$\int_C \mathbf{F} \cdot\, d\mathbf{r} = \iint_S \operatorname{curl} \mathbf{F} \cdot\, d\mathbf{S} = \iint_S \operatorname{curl} \mathbf{F} \cdot \mathbf{n}\, dS$$

(19) The Divergence Theorem:

$$\iiint_E \operatorname{div} \mathbf{F}\, dV = \iint_S \mathbf{F} \cdot\, d\mathbf{S}$$