UNIVERSITY of ROCHESTER

Math 164: Multidimensional Calculus

Midterm Exam 2 Solutions

April 3, 2008

Name (please print legibly): ______________________

University ID Number: ______________________

Please check the box of your instructor:

	Thomas Tucker	MWF	1000-1050
	Andrew Ledoan	MWF	0900-0950

- Calculators, cell phones, iPods, and other electronics are not allowed on this exam.
- Please show all your work. You may use the backs of pages if necessary. A correct answer with no work shown will not receive full credit. Please label and circle your final answers.
- You are responsible for checking that this exam has all 8 pages. Please tell us immediately if your exam is missing a page. Missing pages will not contribute to your total score.

Question	Points	Score
1	15	
2	15	
3	12	
4	12	
5	15	
6	16	
7	15	
Total:	100	

1. (15 points) Find the maximum and minimum values of the function $f(x, y) = 3x + 3y + 5$ on the ellipsoid $x^2 + 2y^2 = 24$.

Solution.

Using the technique of Lagrange multipliers we find that if f has a max or min at (x, y), then there is a real number λ such that

$$\begin{aligned} 3 &= \lambda 2x \\ 3 &= \lambda 4y. \end{aligned} \tag{1}$$

This means $\lambda 4y = \lambda 2x$. Since $\lambda \neq 0$ (otherwise, (1) would not hold), this becomes

$$x = 2y.$$

Plugging $x = 2y$ into $x^2 + 2y^2 = 24$, we obtain

$$4y^2 + 2y^2 = 24$$

so $y = \pm 2$. When $x = -2$, we have $y = -4$, so

$$f(x, y) = 3(-2) + 3(-4) + 5 = -13.$$

When $x = 2$, we have $y = 4$, so

$$f(x, y) = 3(2) + 3(4) + 5 = 23.$$

Thus,

$$23 \text{ is the maximum}$$

and

$$-13 \text{ is the minimum}$$

of $f(x, y)$ on this ellipsoid.

2. (15 points) Find the points(s) on the sphere $x^2 + y^2 + z^2 = 9$ on which the tangent plane is parallel to the plane $x + 2y + 2z = 11$.

Solution. Let S be the surface given by $F(x, y, z) = x^2 + y^2 + z^2$. Then $F_X(x, y, z) = 2x$, $F_y(x, y, z) = 2x$, and $F_z(x, y, z) = 2z$. Suppose that (x_0, y_0, z_0) lies on S. Then

$$2x_0\mathbf{i} + 2y_0\mathbf{j} + 2z_0\mathbf{k} = k\mathbf{i} + 2k\mathbf{j} + 2k\mathbf{k}.$$

(Since the two planes are parallel, the normal vector of one plane is a scalar multiple of the other.) Thus $x_0 = k/2$, $y_0 = k$, and $z_0 = k$. Thus $x_0^2 + y_0^2 + z_0^2 = 9$ implies that $(k/2)^2 + k^2 + k^2 = 9$, or $9k^2/4 = 9$, or $k^2 = 4$. This gives $k = \pm 2$. Hence the required points are $(\pm 1, \pm 2, \pm 2)$.

3. (12 points) Evaluate $\iint_D xe^{x^3}\,dA$, where $D = \{(x, y) \mid 0 \le y \le 1, y \le x \le 1\}$.

Solution. Write

$$\begin{aligned}
\iint_D xe^{x^3}\,dA &= \int_0^1 \int_0^x xe^{x^3}\,dy\,dx \\
&= \int_0^1 \left[yxe^{x^3}\right]_{y=0}^{y=x} dx \\
&= \int_0^1 x^2e^{x^3}\,dx \\
&= \left[\frac{e^{x^3}}{3}\right]_{x=0}^{x=1} \\
&= \frac{e}{3} - \frac{1}{3} \\
&= \frac{1}{3}(e-1).
\end{aligned}$$

4. (12 points) Evaluate $\iint_R x\cos(xy)\,dA$, where $R = \{(x, y) \mid 0 \le x \le \pi, 1 \le y \le 2\}$.

Solution. Write

$$\begin{aligned}
\iint_R x\cos(xy)\,dA &= \int_0^\pi \int_1^2 x\cos(xy)\,dy\,dx \\
&= \int_0^\pi x \int_1^2 \cos(xy)\,dy\,dx \\
&= \int_0^\pi x\left[\frac{1}{x}\sin(xy)\right]_{y=1}^{y=2} dx \\
&= \int_0^\pi [\sin(xy)]_{y=1}^{y=2}\,dx \\
&= \int_0^\pi [\sin(2x) - \sin x]\,dx \\
&= \left[-\frac{1}{2}\cos(2x) + \cos x\right]_0^\pi \\
&= \left[-\frac{1}{2}\cos(2x) + \cos x\right]_0^\pi \\
&= \frac{1}{2} - 1 + \frac{1}{2} - 1 \\
&= -2.
\end{aligned}$$

5. (15 points) Find the maximum rate of change of the function $f(x, y) = x^2 e^{-y}$ at the point $(2, 0)$ and the direction in which it occurs.

Solution. Recall that the maximum rate of change of a function f at a point (a, b) is simply $|\nabla f(a, b)|$ and that the direction in which this is obtained is simply the direction of $\nabla f(a, b)$.

In this case, we have $\nabla f = \langle 2xe^{-y}, -x^2 e^{-y} \rangle$, so

$$\nabla f(2, 0) = \langle 4, -4 \rangle.$$

Thus, the maximum rate of change is

$$|\langle 4, -4 \rangle| = \sqrt{4^2 + 4^2} = \sqrt{32} = 4\sqrt{2}$$

and the direction in which it occurs is

$$\frac{1}{2\sqrt{2}} \langle 4, -4 \rangle = \left\langle \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \right\rangle.$$

6. (16 points) Find and classify, as local maximum, local minimum, or saddle point, the critical points of the function $f(x, y) = x^3 - 3x + 2y^3 - 24y^2$.

Solution. We have $f_x(x, y) = 3x^2 - 3$ and $f_x(x, y) = 6y^2 - 48y$. Note that $f_x(x, y) = 0$ if and only if $3x^2 - 3 = 0$ if and only if $3(x + 1)(x - 10$ if and only if $x = -1$ or $x = 1$. Similarly, $f_y(x, y) = 0$ if and only if $6y(y - 8) = 0$ if and only if $y = 0$ or $y = 8$. Thus the four critical poits are $(1, 8)$, $(1, 0)$, $(-1, 8)$, and $(-1, 0)$.

According to the Second Derivative Test, the discriminant is

$$D = f_{xx}(x, y)f_{yy}(x, y) - f_{xy}^2(x, y).$$

However

$$f_{xx}(x, y) = 6x, \quad f_{yy}(x, y) = 12y - 48, \quad f_{xy}(x, y) = 0.$$

Hence at $(1, 8)$ we have $D > 0$ $f_{xx} > 0$, so that $(1, 8)$ is a local minimum. At $(1, 0)$ we have $D < 0$ and $f_{xx}(x, y) > 0$, so that $(1, 0)$ is a saddle point. (See Note 1 and Note 2 on pages 954 and 955.) At $(-1, 8)$ we have $D < 0$ and $f_{xx}(x, y) < 0$, so that $(-1, 8)$ is a saddle point. Lastly, at $(-1, 0)$ we have $D > 0$ and $f_{xx}(x, y) < 0$, so that $(-1, 0)$ is a local maximum.

7. (15 points)

(a) **(7 points)** Let $w = e^{x+y}$ where $x = \sin t$, $y = tu^2$. Find the numerical values of $\partial w/\partial t$ and $\partial w/\partial u$ when $(t, u) = (\pi, 2)$.

Solution. Using the chain rule, we have

$$\frac{\partial w}{\partial t} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial t} = e^{\sin t + tu^2}\cos t + e^{\sin t + tu^2}u^2 \tag{2}$$

Pluggin in $t = \pi$ and $u = 2$, we obtain

$$\frac{\partial w}{\partial t}(\pi, 2) = -e^{4\pi} + 4e^{4\pi} = 3e^{4\pi}.$$

Similarly, we have

$$\frac{\partial w}{\partial u} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial u} = e^{\sin t + tu^2}(0) + e^{\sin t + tu^2}2tu \tag{3}$$

Pluggin in $t = \pi$ and $u = 2$, we obtain

$$\frac{\partial w}{\partial u}(\pi, 2) = 4\pi e^{4\pi}.$$

(b) **(8 points)** The width of a rectangle is increasing at a rate of 2 in/s., while its length is decreasing at a rate of 1 in/s. At what rate is the area of the rectangle changing when the length is 5 in and and width is 3 in.

Solution.

Since the formual for area is $A(\ell, w) = \ell w$, we have

$$\begin{aligned}\frac{\mathrm{d}A}{\mathrm{d}t} &= \frac{\mathrm{d}A}{\mathrm{d}w}\frac{\mathrm{d}w}{\mathrm{d}t} + \frac{\mathrm{d}A}{\mathrm{d}\ell}\frac{\mathrm{d}\ell}{\mathrm{d}t} \\ &= \ell\frac{\mathrm{d}w}{\mathrm{d}t} + w\frac{\mathrm{d}\ell}{\mathrm{d}t} \\ &= 5(2) + 3(-1) = 7 \text{ square inches per second.}\end{aligned} \tag{4}$$