Math 164: Multivariable Calculus

Midterm Exam 2 ANSWERS

December 3, 2012

1. (15 points)

Find the extremum points and the maximum and minimum values of the function $f(x, y) = e^{-xy}$ subject to the constraint $x^2 - xy + y^2 = 9$.

Answer:

Let $g(x, y) = x^2 - xy + y^2$. We would like to find values x, y, and λ such that $\nabla f = \lambda \nabla g$. That is, we would like to solve the system

$$-ye^{-xy} = \lambda(2x - y) \tag{1}$$
$$-xe^{-xy} = \lambda(-x + 2y) \tag{2}$$
$$x^2 - xy + y^2 = 9 \tag{3}$$

Assume x and y are both nonzero. Since e^{-xy} is never zero, this implies that $\lambda \neq 0$ as well. Solving for $\frac{-e^{-xy}}{\lambda}$ in the first two equations and setting the expressions equal to each other gives us

$$\frac{2x - y}{y} = \frac{-x + 2y}{x}.$$

This implies

$$2x^2 - xy = -xy + 2y^2.$$

So

$$x^2 = y^2.$$

So $x = \pm y$. If $x = y$, equation (3) implies that $x^2 = 9$, that is $x = \pm 3$. If $x = -y$, equation (3) implies that $3x^2 = 9$, that is $x = \pm\sqrt{3}$.

Assume $x = 0$. Then equation (3) implies that $y = \pm 3$. (Note $\lambda = 0$ in this case.)

Assume $y = 0$. Then equation (3) implies that $x = \pm 3$. (Note $\lambda = 0$ in this case.)

So the list of potential extreme points for f is $\pm(3, 3)$, $\pm(\sqrt{3}, -\sqrt{3})$, $\pm(0, 3)$, $\pm(3, 0)$. We plug these into f to determine the answer, which follows.

The function f has a maximum value of e^3 at $\pm(\sqrt{3}, -\sqrt{3})$, and a minimum value of e^{-9} at $\pm(3, 3)$.

2. (15 points)

Consider the double integral

$$\int_0^{\pi/4} \int_0^{1/\cos\theta} r^3 \, dr \, d\theta.$$

(a) Sketch the region of integration.

Answer:

Note that $0 \leq \theta \leq \frac{\pi}{4}$ and $0 \leq r \leq \frac{1}{\cos\theta}$. This latter inequality implies that $r\cos\theta \leq 1$, that is, $x \leq 1$. Therefore, the region is the triangular region in the first quadrant bound by $y = 0$, $y = x$, and $x = 1$.

(b) Convert the above integral to rectangular coordinates.

Answer:

The triangular region from part (a) can be described as $D = \{(x, y) : 0 \leq y \leq x, 0 \leq x \leq 1\}$. Recall that $r^2 = x^2 + y^2$. Also, $r\,dr\,d\theta$ is the area element dA. So we have

$$\int_0^{\pi/4} \int_0^{1/\cos\theta} r^3 \, dr \, d\theta = \int_0^{\pi/4} \int_0^{1/\cos\theta} r^2 r \, dr \, d\theta = \iint_D r^2 \, dA = \int_0^1 \int_0^x (x^2 + y^2) \, dy \, dx.$$

(c) Change the order of integration in rectangular coordinates.

Answer:

To change the order of integration in our answer to part (b), we'd like the area element dA to be $dx\,dy$. To do this, describe the triangular region of integration as $D = \{(x, y) : y \leq x \leq 1, 0 \leq y \leq 1\}$. Then we have

$$\int_0^{\pi/4} \int_0^{1/\cos\theta} r^3 \, dr \, d\theta = \int_0^1 \int_y^1 (x^2 + y^2) \, dx \, dy.$$

(d) Evaluate the integral in any form you desire.

Answer:

We use our answer to part (b):

$$\begin{aligned}\int_0^{\pi/4}\int_0^{1/\cos\theta} r^3\,dr\,d\theta &= \int_0^1\int_0^x (x^2+y^2)\,dy\,dx\\ &= \int_0^1 \left(x^2y+\frac{y^3}{3}\right)\Big|_{y=0}^{y=x} dx\\ &= \int_0^1 \left(\frac{4}{3}x^3\right) dx\\ &= \frac{4}{3}\left(\frac{x^4}{4}\Big|_0^1\right)\\ &= \frac{1}{3}\end{aligned}$$

3. (15 points)

Consider a lamina that occupies the region in the xy-plane bound by $x = 0$, $y = 0$ and the parabola $y = 1 - x^2$, where $x \geq 0$. The lamina has density function $\rho(x, y) = x + 1$.

(a) Find the mass of the lamina.

Answer:

The region of integration is $D = \{(x, y) : 0 \leq y \leq 1 - x^2, 0 \leq x \leq 1\}$. The mass of the lamina is

$$\begin{aligned}m = \iint_D \rho(x,y)\,dA &= \int_0^1\int_0^{1-x^2} (x+1)\,dy\,dx\\ &= \int_0^1 (x+1)y\big|_{y=0}^{y=1-x^2}\,dx = \int_0^1 (x+1)(1-x^2)\,dx\\ &= \int_0^1 (x - x^3 + 1 - x^2)\,dx = \left(\frac{x^2}{2} - \frac{x^4}{4} + x - \frac{x^3}{3}\right)\Big|_0^1\\ &= \frac{1}{2} - \frac{1}{4} + 1 - \frac{1}{3} = \frac{11}{12}\end{aligned}$$

(b) Find the x-coordinate of the center of mass of the lamina.

Answer:

$$\begin{aligned}
\overline{x} &= \frac{1}{m}\iint_D x\rho(x,y)\,dA = \frac{12}{11}\int_0^1\int_0^{1-x^2} x(x+1)\,dy\,dx \\
&= \frac{12}{11}\int_0^1 (x^2+x)y\Big|_{y=0}^{y=1-x^2}\,dx = \frac{12}{11}\int_0^1 (x^2+x)(1-x^2)\,dx \\
&= \frac{12}{11}\int_0^1 (x^2-x^4+x-x^3)\,dx = \frac{12}{11}\left(\frac{x^3}{3}-\frac{x^5}{5}+\frac{x^2}{2}-\frac{x^4}{4}\right)\Bigg|_0^1 \\
&= \frac{12}{11}\left(\frac{1}{3}-\frac{1}{5}+\frac{1}{2}-\frac{1}{4}\right) = \frac{23}{55}
\end{aligned}$$

4. (10 points)

Let B be the ball $x^2+y^2+z^2 \leq a^2$ of radius a. Its volume $\dfrac{4\pi a^3}{3}$ can be computed as a triple integral $\iiint_B dV$. Set up iterated integrals for computing this triple integral in

(a) rectangular coordinates.

Answer:

$$\int_{-a}^{a}\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}}\int_{-\sqrt{a^2-x^2-y^2}}^{\sqrt{a^2-x^2-y^2}} dz\,dy\,dx$$

(b) cylindrical coordinates.

Answer:

$$\int_0^{2\pi}\int_0^{a}\int_{-\sqrt{a^2-r^2}}^{\sqrt{a^2-r^2}} r\,dz\,dr\,d\theta$$

(c) spherical coordinates.

Answer:

$$\int_0^{2\pi}\int_0^{\pi}\int_0^{a} \rho^2\sin\phi\,d\rho\,d\phi\,d\theta$$

5. (15 points)

Evaluate the triple integral by converting to cylindrical coordinates:

$$\int_0^1 \int_0^{\sqrt{1-x^2}} \int_1^{2-\sqrt{x^2+y^2}} \frac{xy}{(x^2+y^2)\sqrt{x^2+y^2}}\, dz\, dy\, dx$$

Answer:

$$\begin{aligned}
&\int_0^1 \int_0^{\sqrt{1-x^2}} \int_1^{2-\sqrt{x^2+y^2}} \frac{xy}{(x^2+y^2)\sqrt{x^2+y^2}}\, dz\, dy\, dx \\
&= \int_0^{\pi/2} \int_0^1 \int_1^{2-r} \frac{(r\cos\theta)(r\sin\theta)}{r^2 r} r\, dz\, dr\, d\theta \\
&= \int_0^{\pi/2} \int_0^1 \int_1^{2-r} \cos\theta \sin\theta\, dz\, dr\, d\theta \\
&= \int_0^{\pi/2} \int_0^1 \cos\theta \sin\theta \left(z\big|_1^{2-r} \right) dr\, d\theta \\
&= \int_0^{\pi/2} \cos\theta \sin\theta\, d\theta \int_0^1 (1-r)\, dr \\
&= \frac{\sin^2\theta}{2}\bigg|_0^{\pi/2} \left(r - \frac{r^2}{2}\right)\bigg|_0^1 \\
&= \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}
\end{aligned}$$

6. (15 points)

Let R be the triangular region in the xy-plane whose vertices have coordinates $(-1,-3), (3,1)$ and $(0,0)$. Let T be the transformation from the uv-plane to the xy-plane given by

$$T(u,v) = (3u+v, 3v+u).$$

(a) There are three points in the uv-plane whose images under the transformation are the vertices of the triangle. What are these three points?

Answer:

For each given point (x,y), we need to find (u,v) such that $T(u,v) = (x,y)$. Note that $T(0,0) = (0,0)$, $T(0,-1) = (-1,-3)$, and $T(1,0) = (3,1)$. Therefore the three points asked for are $(0,0)$, $(0,-1)$, and $(1,0)$.

(b) What is the Jacobian of this transformation?

Answer:

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 3 & 1 \\ 1 & 3 \end{vmatrix} = 9 - 1 = 8$$

(c) Evaluate the following double integral:

$$\iint_R (x - 3y)\, dA$$

Answer:

Let D denote the region in the uv-plane such that $T(D) = R$. From part (a), we know that this is the triangle in the uv-plane bounded by the lines $u = 0$, $v = 0$, and $v = u - 1$. That is, $D = \{(u, v) : u - 1 \leq v \leq 0, 0 \leq u \leq 1\}$. Using change of variables,

$$\begin{aligned}
\iint_R (x - 3y)\, dA &= \int_0^1 \int_{u-1}^0 (3u + v - 3(3v + u)) \left| \frac{\partial(x,y)}{\partial(u,v)} \right| dv\, du \\
&= -\int_0^1 \int_0^{u-1} (-8v)(8)\, dv\, du \\
&= 64 \int_0^1 \int_0^{u-1} v\, dv\, du \\
&= 64 \int_0^1 \left. \frac{v^2}{2} \right|_0^{u-1} du \\
&= 32 \int_0^1 (u^2 - 2u + 1)\, du \\
&= 32 \left. \left(\frac{u^3}{3} - u^2 + u \right) \right|_0^1 \\
&= \frac{32}{3}
\end{aligned}$$

7. (15 points)

Evaluate

$$\iiint_{\mathcal{B}} z e^{(x^2+y^2+z^2)^2}\, dV$$

where $\mathcal{B}$ is the hemispherical solid bounded by the sphere of radius 2 centered at the origin and the xy-plane, with $z \geq 0$.

Answer:

$$\begin{aligned}
\iiint\limits_{\mathcal{B}} z e^{(x^2+y^2+z^2)^2}\, dV &= \int_0^{2\pi} \int_0^{\pi/2} \int_0^2 (\rho\cos\phi) e^{\rho^4} \rho^2 \sin\phi \, d\rho\, d\phi\, d\theta \\
&= \int_0^{2\pi} \int_0^{\pi/2} \int_0^2 \rho^3 e^{\rho^4} \cos\phi \sin\phi \, d\rho\, d\phi\, d\theta \\
&= \int_0^{2\pi} d\theta \int_0^{\pi/2} \cos\theta \sin\theta \, d\phi \int_0^2 \rho^3 e^{\rho^4}\, d\rho \\
&= \left(\theta\big|_0^{2\pi}\right) \left(\frac{\sin^2\theta}{2}\bigg|_0^{\pi/2} \right) \left(\frac{1}{4} e^{\rho^4} \bigg|_0^2 \right) \\
&= (2\pi) \left(\frac{1}{2}\right) \left(\frac{1}{4}\left(e^{16} - 1\right) \right) \\
&= \frac{\pi}{4}(e^{16} - 1)
\end{aligned}$$