MATH 164

Midterm 1 ANSWERS

October 18, 2011

1. (12 points)

(a) Find the angle between vectors $\mathbf{a} = (1, 2, 3)$ and $\mathbf{b} = (-1, 2, -2)$ in terms of an inverse trig function. Simplify as much as you can without using a calculator.

(b) Find a vector which is perpendicular to the two vectors above.

(c) Find a unit vector which is parallel to $(-1, 2, -2)$.

Answer:

(a) If θ is the angle between the vectors $\mathbf{a}$ and $\mathbf{b}$, then $\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}| \cdot |\mathbf{b}| \cos\theta$. Since $\mathbf{a} \cdot \mathbf{b} = -1 + 4 - 6 = -3$, $|\mathbf{a}| = \sqrt{1+4+9} = \sqrt{14}$, $|\mathbf{b}| = \sqrt{1+4+4} = 3$, we have

$$\theta = \arccos\left(\frac{-3}{\sqrt{14} \cdot 3}\right) = \arccos\left(\frac{-1}{\sqrt{14}}\right)$$

(b) We can take the cross product $\mathbf{a} \times \mathbf{b}$, which is perpendicular to both $\mathbf{a}$ and $\mathbf{b}$. Expressing the vectors in terms of $\mathbf{i}, \mathbf{j}, \mathbf{k}$, we get

$$\begin{aligned}(\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}) \times (-\mathbf{i} + 2\mathbf{j} - 2\mathbf{k}) &= 2\mathbf{k} + 2\mathbf{j} + 2\mathbf{k} - 4\mathbf{i} - 3\mathbf{j} - 6\mathbf{i} \\ &= -10\mathbf{i} - \mathbf{j} + 4\mathbf{k}\end{aligned}$$

(c) Since $|(-1, 2, -2)| = 3$ by part (a), we need only divide the vector by 3. This gives

$$\left(-\frac{1}{3}, \frac{2}{3}, -\frac{2}{3}\right)$$

The negative of this vector will also work.

2. (13 points)

Let $\mathbf{r}(t)$ denote the curve with parametric equation

$$x(t) = t^2, \qquad y(t) = 2\sin(t) - 1, \qquad z(t) = 2\cos(t)$$

(a) Compute the velocity vector, the speed and the acceleration vector.

(b) The curve $\mathbf{r}(t)$ lies on a cylinder. Find the equation of that cylinder.

Answer:

(a)

$$\begin{aligned}\mathbf{v}(t) &= (2t, 2\cos(t), -2\sin(t)) \\ v(t) &= ||\mathbf{v}(t)|| = 2\sqrt{t^2+1} \\ \mathbf{a}(t) &= (2, -2\sin(t), -2\cos(t))\end{aligned}$$

(b) One can notice that $(y(t)+1)^2 + z(t)^2 = 4$, so the equation of the cylinder is given by

$$(y+1)^2 + z^2 = 4 \qquad x \in \mathbb{R}.$$

This is a cylinder of radius 2, whose center line is parallel to the x-axis and intersecting the (y, z)-plane through the point (0,-1,0).

3. (13 points)

(a) Find an integral which gives the arc length of the curve between $t = 0$ and $t = 1$. **Do not solve the integral**.

$$r(t) = (t^2+1, e^{t^2}, \sqrt{t})$$

(b) Find the curvature $\kappa(t)$ of the curve

$$r(t) = (t^2+1, t^3, t)$$

Answer:

(a) The formula is $L = \int_a^b |r'(t)|dt$. Therefore in our case,

$$\begin{aligned}L &= \int_0^1 \left[(2t)^2 + (e^{t^2}2t)^2 + \left(\frac{1}{2\sqrt{t}}\right)^2\right]^{1/2} dt \\ &= \int_0^1 \left[4t^2 + 4t^2e^{2t^2} + \frac{1}{4t}\right]^{1/2} dt\end{aligned}$$

(b) We use the formula

$$\kappa(t) = \frac{|r'(t) \times r''(t)|}{|r'(t)|^3}$$

Using $\mathbf{i}, \mathbf{j}, \mathbf{k}$ notation, we have

$$\begin{aligned} r'(t) &= 2t\mathbf{i} + 3t^2\mathbf{j} + \mathbf{k} \\ r''(t) &= 2\mathbf{i} + 6t\mathbf{j} \end{aligned}$$

Then

$$\begin{aligned} |r'(t)| &= \left[(2t)^2 + (3t^2)^2 + 1\right]^{1/2} = \left[9t^4 + 4t^2 + 1\right]^{1/2} \\ r'(t) \times r''(t) &= 12t^2\mathbf{k} - 6t^2\mathbf{k} + 2\mathbf{j} - 6t\mathbf{i} = -6t\mathbf{i} + 2\mathbf{j} + 6t^2\mathbf{k} \\ |r'(t) \times r''(t)| &= \left[(-6t)^2 + 2^2 + (6t^2)^2\right]^{1/2} = \left[36t^4 + 36t^2 + 4\right]^{1/2} = 2[9t^4 + 9t^2 + 1]^{1/2} \end{aligned}$$

Finally,

$$\kappa(t) = \frac{|r'(t) \times r''(t)|}{|r'(t)|^3} = \frac{2[9t^4 + 9t^2 + 1]^{1/2}}{[9t^4 + 4t^2 + 1]^{3/2}}$$

4. (13 points)

Consider the function

$$f(x, y) = x^2 - 6x + y^2 - 2y + 10$$

(a) What type of quadric surface is the graph $z = f(x, y)$?

(b) Find an equation of the tangent plane to the graph when $x = 5$ and $y = -1$

(c) Consider the function $h : \mathbb{R}^2 \to \mathbb{R}$ defined as $h(x, y) = \sqrt{4 - ||\nabla f(x, y)||^2}$. What is the domain of h ?

Answer:

(a) Since $z = (x - 3)^2 + (y - 1)^2$ the graph has to be a circular paraboloid, with the axis in the z direction. Its tip lies in the (x, y)-plane, at the point (3,1,0).

(b) The equation of the tangent plan is given by

$$z = f(5, -1) + f_x(5. - 1)(x - 5) + f_y(5, -1)(y + 1)$$

The partial derivatives are $f_x(x, y) = 2x - 6$ and $f_y(x, y) = 2y - 2$, hence the equation of the tangent plane becomes:

$$z = 8 + 4(x - 5) + (-4)(y + 1) \qquad \text{or} \qquad 4x - 4y - z = 16$$

(c)

$$\nabla f(x,y) = (f_x, f_y) = (2x-6, 2y-2) \Longrightarrow h(x,y) = 2\sqrt{1-(x-3)^2-(y-1)^2}$$

h can only be defined on a domain where the square root exists, so one must have that

$$1-(x-3)^2-(y-1)^2 \geq 0.$$

This gives

$$(x-3)^2+(y-1)^2 \leq 1,$$

so the domain is the closed disk, centered at $(3,1)$, i.e.

$$\mathcal{D} = \{(x,y) \in \mathbb{R}^2 | (x-3)^2+(y-1)^2 \leq 1\}.$$

5. (13 points)

One of the following functions has a limit as $(x,y) \to (0,0)$. Find the limit if it exists, or state that it doesn't exist. In either case, you must explain your reasoning to get full credit.

(a)

$$f(x,y) = \frac{x^2 y}{x^4+y^6}$$

(b)

$$f(x,y) = \frac{x^4+y^5}{x^2+y^4}$$

Answer:

(a) The limit does not exist. If we let $x = y$ then

$$\frac{x^2 y}{x^4+y^6} = \frac{y^3}{y^4+y^6} = \frac{1}{y+y^3}$$

and the denominator tends to 0 while the numerator is fixed.

(b) The limit exists, and equals 0. We can break up the fraction as follows,

$$\frac{x^4+y^5}{x^2+y^3} = \frac{x^4}{x^2+y^4} + \frac{y^5}{x^2+y^4}$$

We have

$$\left|\frac{x^4}{x^2+y^4}\right| \leq \left|\frac{x^4}{x^2}\right| = |x^2|$$

which tends to 0 as $x \to 0$. For the second term, we have

$$\left|\frac{y^5}{x^2+y^4}\right| \le \left|\frac{y^5}{y^4}\right| = |y|$$

which tends to 0 as $y \to 0$.

6. (12 points)

(a) Find $\mathbf{r}(t)$ if $\mathbf{r}'(t) = t\mathbf{i} + e^t\mathbf{j} + te^t\mathbf{k}$ and $\mathbf{r}(0) = \mathbf{i} + \mathbf{j} + \mathbf{k}$.

(b) If $\mathbf{r}(t) = (e^{2t}, e^{-2t}, te^{2t})$ find $\mathbf{r}'(t)$ and $\mathbf{r}'(t) \cdot \mathbf{r}''(t)$.

Answer:

(a)

$$\begin{aligned} \mathbf{r}(t) &= \int \mathbf{r}'(t)\ dt = \int \left(t\mathbf{i} + e^t\mathbf{j} + te^t\mathbf{k}\right)\ ds \\ &= \frac{t^2}{2}\mathbf{i} + e^t\mathbf{j} + (te^t - e^t)\mathbf{k} + \mathbf{C} \end{aligned}$$

But $\mathbf{r}(0) = \mathbf{i} + \mathbf{j} + \mathbf{k}$, which means that $\mathbf{C} = \mathbf{i} + 2\mathbf{k}$. So the curve is: $\mathbf{r}(t) = \left(\frac{t^2}{2} + 1\right)\mathbf{i} + e^t\mathbf{j} + \left(te^t - e^t + 2\right)\mathbf{k}$.

(b)

$$\begin{aligned} \mathbf{r}(t) &= (e^{2t}, e^{-2t}, te^{2t}) \\ \mathbf{r}'(t) &= (2e^{2t}, -2e^{-2t}, e^{2t} + 2te^{2t}) \\ \mathbf{r}''(t) &= (4e^{2t}, 4e^{-2t}, 4e^{2t} + 4te^{2t}) \\ \mathbf{r}'(t) \cdot \mathbf{r}''(t) &= 8e^{4t} - 8e^{-4t} + 4e^{4t} + 12te^{4t} + 8t^2e^{4t} \\ &= 12e^{4t} - 8e^{-4t} + 12te^{4t} + 8t^2e^{4t} \end{aligned}$$

7. (12 points)

Suppose that $\frac{\partial f}{\partial x}(1,2) = 3$ and $\frac{\partial f}{\partial y}(1,2) = -4$. If $r(t) = (2t^2 - t, -6t + 2)$, find $\frac{df(r(t))}{dt}$ at $t = 1$.

Answer:

The chain rule for functions of several variables states that

$$\frac{df(r(t))}{dt} = \nabla f(r(t)) \cdot r'(t)$$

Note that $r'(t) = (4t - 1, -6)$ and $r'(1) = (3, -6)$. Also $r(1) = (1, 2)$. So we have

$$\nabla f(r(1)) = \left(\frac{\partial f}{\partial x}(1,2), \frac{\partial f}{\partial y}(1,2)\right) = (3, -4)$$

Thus, substituting into the chain rule, we find

$$\frac{df(r(t))}{dt} = (3, -4) \cdot (3, -6) = 9 + 24 = 33$$

8. (12 points) The linear approximation at $(x, y) = (0, 0)$ of $\sqrt{y + \cos^2 x}$ takes the form $a + bx + cy$. Find a, b, c and write the resulting linear approximation.

Answer:

The linear approximation for $f(x, y) = \sqrt{y + \cos^2 x}$ is given by

$$L(x, y) = f(0, 0) + f_x(0, 0)(x - 0) + f_y(0, 0)(y - 0).$$

But $f_x(x, y) = -\sin x \cos x \left(y + \cos^2 x\right)^{-1/2}$ and $f_y(x, y) = \frac{1}{2}\left(y + \cos^2 x\right)^{-1/2}$. So $f(0, 0) = 1$, $f_x(0, 0) = 0$, and $f_y(0, 0) = \frac{1}{2}$.

Thus $L(x, y) = 1 + \frac{1}{2}y$.