MATH 164

PRACTICE FINAL

Tuesday December 16th

4:00pm – 7:00pm
(3 hours)

ROUGH SOLUTIONS

1. Let $P = (1, 0, 0)$, $Q = (0, 1, 0)$, $R = (0, 0, -1)$

a. Find the equation of the plane passing through P, Q and R

b. Find the orthogonal projection of the vector $(1, 2, 3)$ onto the plane passing through P, Q and R. (this means the vector in the plane closest to $(1, 2, 3)$)

c. Find the distance between $(1, 2, 3)$ and the plane passing through P, Q and R.

solution:

a. $x + y - z = 1$ works. To get this solution, notice that $P - R$ and $Q - R$ are parallel to the plane we are looking for, so a normal vector would be $(P - R) \times (Q - R)$. This is $(-1, -1, 1)$ which gives the answer above.

b. We are looking for a vector $\vec{v} = (x, y, z)$ which satisfies the equation $x + y - z = 1$ and $\vec{v} = (1, 2, 3) + \vec{w}$ where $\vec{w}$ is perpendicular to the plane $x + y - z = 1$.

Here is how we compute $\vec{w}$. $\vec{w} = k(1, 1, -1)$ so we compute k. If we stick $(1, 2, 3) + k(1, 1, -1)$ into the equation $x + y - z = 1$ and solve for k we get $k = \frac{1}{3}$ so $\vec{v} = (\frac{4}{3}, \frac{7}{3}, \frac{8}{3})$.

c. This is the length of the vector $\frac{1}{3}(1, 1, -1)$ which is $\frac{1}{\sqrt{3}}$.

2. Find the volume of the parallelepiped determined by the vectors $\vec{i} + \vec{j}$, $\vec{j} + \vec{k}$, $\vec{i} + \vec{k}$.

solution:

this is absolute value of the determinant of the matrix

$$\begin{matrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{matrix}$$

which is 2.

3. Reparametrize the curve $\vec{r}(t)$ with respect to arc-length from the point where $t = 0$.
$\vec{r}(t) = \langle e^t \cos t, e^t \sin t\rangle$

solution:

$|\vec{r}'(t)| = \sqrt{2}e^t$ so $s(t) = \int_0^t |\vec{r}'(t)| = \sqrt{2}(e^t - 1)$ so $t(s) = \ln(\frac{s}{\sqrt{2}} + 1)$ and so

$$\vec{r}(s) = (\frac{s}{\sqrt{2}} + 1)\langle\cos(\ln(\frac{s}{\sqrt{2}} + 1)), \sin(\ln(\frac{s}{\sqrt{2}} + 1))\rangle$$

4. Let $f(x, y) = \frac{x}{y}$.

a. Find the linear approximation of $f(x, y)$ at $(6, 3)$.

b. Approximate $\frac{5.8}{2.9}$ using the above linear approximation.

solution:

a. $z = 2 + \frac{1}{3}(x - 6) - \frac{2}{3}(y - 3)$

b. $(x, y) = (5.8, 2.9)$ so $z = 2 + \frac{1}{3}(-0.2) - \frac{2}{3}(-0.1) = 2$

5. Find and classify the critical points of the function $f(x, y) = 2x^3 + xy^2 + 5x^2 + y^2$

solution:

There are four critical points $(0, 0)$, $(-\frac{5}{3}, 0)$, $(-1, 2)$ and $(-1, -2)$.

The hessian has the form

$$\begin{matrix} 12x + 10 & 2y \\ 2y & 2x + 2 \end{matrix}$$

so

$(0, 0)$ is a local min, $(-\frac{5}{3}, 0)$ is a local max, $(-1, 2)$ is a saddle and $(-1, -2)$ is also a saddle.

6. Find the maximum and minimum values of the function $f(x, y, z) = x^4 + y^4 + z^4$ subject to the constraints $x^2 + y^2 + z^2 = 1$.

solution:

Lagrange multipliers tells us that there are four equations that need to be satisfied:

$$2x^3 = \lambda x$$

$$2y^3 = \lambda y$$

$$2z^3 = \lambda z$$

$$x^2 + y^2 + z^2 = 1$$

There are a total of 26 solutions to these equations. There are 6 solutions that have only one non-zero term, they are

$\{(\pm 1, 0, 0), (0, \pm 1, 0), (0, 0, \pm 1)\}$ here $\lambda = 2$.

There are 12 solutions with two non-zero terms, they are:

$\{\pm \frac{1}{\sqrt{2}}(1, \pm 1, 0), \pm \frac{1}{\sqrt{2}}(1, 0, \pm 1), \pm \frac{1}{\sqrt{2}}(0, 1, \pm 1)\}$ here $\lambda = 1$.

There are 8 solutions with all non-zero terms, they satisfy $\lambda = \frac{2}{3}$ so $(x, y, z) = (\pm \frac{1}{\sqrt{3}}, \pm \frac{1}{\sqrt{3}}, \pm \frac{1}{\sqrt{3}})$

So the maximum value is 1 (which occurs at the first 6 solutions) and the minimum value is $\frac{1}{3}$ (which occurs at the last 8 solutions).

note to students: question 6 is a little hard for an exam, but if you got it, you're certainly ready for any lagrange multipliers question that you'd see on the exam!

7. Compute the volume of the region under the paraboloid $z = x^2 + y^2$ and above the disc $x^2 + y^2 \leq 9$.

solution:

this is the integral of $x^2 + y^2$ over the disc $x^2 + y^2 \leq 9$, which we can compute in polar coordinates

$$\int_0^{2\pi} \int_0^3 r^2 r dr d\theta$$

which is $2\pi \frac{3^4}{4}$

8. Find center of mass of the region D in the plane, where D is bounded by the parabola $y = 9 - x^2$ and the x-axis, with $\rho(x, y) = y$.

solution:

$m = \int_{-3}^{3} \int_{0}^{9-x^2} ydydx = \cdots = 81 + \frac{3^5}{5}$

$(\overline{x}, \overline{y}) = (0, \frac{1}{m} \int_{-3}^{3} \int_{0}^{9-x^2} y^2 dydx)$ (the x coordinate is zero by symmetry)

$\cdots = \frac{1}{m}(0, \frac{2}{3} \left(3 \cdot 9^3 - 81 \cdot 3^3 + \frac{27}{5} 3^5 - \frac{3^7}{7}\right))$

9. Evaluate $\int \int \int_E zdV$ where E lies between the spheres $x^2+y^2+z^2 = 1$ and $x^2+y^2+z^2 = 4$ in the first octant.

solution:

In spherical polar coordinates this is the integral

$$\int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \int_1^2 \rho \cos(\phi) \rho^2 \sin(\phi) d\rho d\phi d\theta = \cdots = \frac{15\pi}{16}$$

10. Consider the vector field $\vec{F}(x, y, z) = \langle yze^{xz}, e^{xz}, xye^{xz} \rangle$

a. Is $\vec{F}$ conservative? If so, find a potential function for $\vec{F}$.

b. What is the integral $\int_C \vec{F} \cdot d\vec{r}$ where C is the curve that satisfies the equations $x^2 + \frac{y^2}{2} = 1$ and $x + y + z = 0$. (orient the curve so that when one looks 'down' on the curve from the z-axis, it is oriented in a counter-clockwise fashion.)

solution:

a. $f = ye^{xz}$ works, meaning $\nabla f = \vec{F}$ so $\vec{F}$ is conservative.

b. 0 since C is a closed curve. This follows from the fundamental theorem of calculus.

11. Evaluate the integral $\int_C e^y dx + 2xe^y dy$ where C is the square with sides $x = 1$, $x = 3$, $y = 1$, $y = 4$ (oriented CW).

solution:

We use Green's theorem. $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = e^y$ so we need to compute the integral $-\int_1^3 \int_1^4 e^y dydx$. Note that the minus sign appears since C is oriented clockwise (CW='clockwise' while CCW='counterclockwise'. clockwise is not the boundary orientation). So the answer is $-2(e^4 - e)$.

12. Let $\vec{F}(x, y, z) = \langle y, z, x \rangle$. Compute the flux integral $\int \int_S \vec{F} \cdot d\vec{S}$ where S is the triangle with vertices $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$. Orient S so that the components of the normal vector are all positive.

solution:

The triangle sits on the plane $x+y+z=1$ so the unit normal vector $\vec{n} = \frac{1}{\sqrt{3}}(1,1,1)$. We parametrize the triangle as the graph of $g(x,y) = 1-x-y$ so $\vec{r}(x,y) = (x,y,1-x-y)$ and $\frac{\partial \vec{r}}{\partial x} \times \frac{\partial \vec{r}}{\partial y} = (1,1,1)$ and our parametrization has parameter-domain $\{x \geq 0, y \geq 0, x+y \leq 1\}$ so the answer is the integral:

$$\int_0^1 \int_0^{1-x} \langle y, 1-x-y, x\rangle \cdot \langle 1,1,1\rangle dy dx = \cdots = \frac{1}{2}$$

13. Let the surface S be the graph of the function $z = x^2 - y^2$ with domain $D = \{(x,y) : -1 \leq x \leq 1, -1 \leq y \leq 1\}$. Give S an 'upward' orientation. Let $\vec{F}$ be the vector-field $\vec{F}(x,y,z) = \langle z, x, y\rangle$ Compute $\int\int_S \vec{F} \cdot d\vec{S}$ (the flux of $\vec{F}$ across the surface S).

solution:

our parametrization is $\vec{r}(x,y) = \langle x, y, x^2-y^2\rangle$ so $\frac{\partial \vec{r}}{\partial x} \times \frac{\partial \vec{r}}{\partial y} = \langle -\frac{\partial g}{\partial x}, -\frac{\partial g}{\partial y}, 1\rangle$ where $g(x,y) = x^2 - y^2$ giving the integral

$$\int_{-1}^1 \int_{-1}^1 \langle x^2 - y^2, x, y\rangle \cdot \langle -2x, 2y, 1\rangle dx dy = \cdots = 0$$

14. Compute the flux integral

$$\int\int_S \vec{F} \cdot d\vec{S}$$

where $\vec{F} = \langle -xz, z, z^2\rangle$ and S is the outwardly oriented boundary of the solid from problem 9.

solution:

by Gauss's divergence theorem this is the integral over the solid E from problem 9:

$$\int\int\int_E \nabla \cdot \vec{F} dV$$

since $\nabla \cdot \vec{F} = z$ the above integral is $\frac{15\pi}{16}$.

15. Verify Stoke's Theorem for the vector field $\vec{F}(x,y,z) = \langle 3y, 4z, -6x\rangle$ and the surface S which is the part of the paraboloid $z = 9 - x^2 - y^2$ over the xy-plane, oriented upward.

solution:

$\nabla \times \vec{F} = \langle -4, 6, -3\rangle$ so the flux integral we need to compute is

$$\int\int_S \langle -4, 6, -3\rangle \cdot d\vec{S}$$

and we need to verify that this is the same as the integral

$$\int_{\partial S} 3ydx + 4zdy - 6xdz$$

to compute the flux integral notice we have the parametrization $\vec{r}(x, y) = (x, y, 9-x^2-y^2)$ so $\frac{\partial \vec{r}}{\partial x} \times \frac{\partial \vec{r}}{\partial y} = \langle -\frac{\partial g}{\partial x}, -\frac{\partial g}{\partial y}, 1\rangle$ where $g(x, y) = 9 - x^2 - y^2$ so our integral is

$$\int\int_D \langle -4, 6, -3\rangle \cdot \langle 2x, 2y, 1\rangle dA$$

where D is the disc of radius 3. If we write the above integral as

$$\int\int_D -8xdA + \int\int_D 12ydA + \int\int_D -3dA$$

clearly the first two integrals are zero by symmetry, and the last integral is $-3\pi 3^2$.

The path integral is much easier to compute. An orientation-preserving parametrization of ∂S is $\vec{r}(t) = 3(\cos(t), \sin(t), 0)$ so the above integral becomes

$$\int_0^{2\pi} 9\sin(t)(-3\sin(t))dt = \cdots = -27\pi$$