Math 164: Multidimensional Calculus

Final Exam

December 15, 2015

NAME (please print legibly): Solutions

Your University ID Number: ____________

Indicate your instructor with a check in the appropriate box:

Bobkova	TR 12:30-1:45pm	
Chen	MW 3:25-4:40pm	
Dummit	TR 3:25-4:40pm	
Salur	MWF 09:00-09:50am	

- You are responsible for checking that this exam has all 19 pages.
- No calculators, phones, electronic devices, books, notes are allowed during the exam.
- Show all work and justify all answers. Please sign the pledge below.

Pledge of Honesty

I affirm that I will not give or receive any unauthorized help on this exam, and that all work will be my own.

Signature: ____________

Part A		
QUESTION	VALUE	SCORE
1	14	
2	12	
3	10	
4	10	
5	12	
6	16	
7	16	
8	10	
TOTAL	100	

Part B		
QUESTION	VALUE	SCORE
9	14	
10	14	
11	12	
12	12	
13	12	
14	12	
15	12	
16	12	
TOTAL	100	

Part A

1. **(14 points)** Consider the two planes $4x + y - z = 4$ and $x + 4y - z = 1$.

(a) Find the (acute) angle between the planes.

$\vec{n_1} = \langle 4, 1, -1 \rangle$

$\vec{n_2} = \langle 1, 4, -1 \rangle$

$\vec{n_1} \cdot \vec{n_2} = |\vec{n_1}| \cdot |\vec{n_2}| \cdot \cos\theta$

$4 + 4 + 1 = \sqrt{4^2 + 1^2 + (-1)^2} \cdot \sqrt{1^2 + 4^2 + (-1)^2} \cdot \cos\theta$

$\frac{9}{18} = \cos\theta$

$\frac{1}{2} = \cos\theta$

$\theta = 60°$

(b) Find a parametrization for the line of intersection of these two planes.

$$\vec{n_1} \times \vec{n_2} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 4 & 1 & -1 \\ 1 & 4 & -1 \end{vmatrix} = 3\hat{\imath} + 3\hat{\jmath} + 15\hat{k}$$

Take a point from the intersection: $x = 0$

$$\begin{cases} y - z = 4 \\ 4y - z = 1 \end{cases}$$

$-y + \cancel{z} = -4$

$4y - \cancel{z} = 1$

$3y = -3$

$y = -1$

$z = -5$

$$\left.\begin{aligned} x(t) &= 0 + 3t \\ y(t) &= -1 + 3t \\ z(t) &= -5 + 15t \end{aligned}\right\}$$ A parametr. for the line of intersection.

2. (12 points) Find the distance between the two skew lines whose parametrizations are

$$l_1 : \langle x, y, z\rangle = \langle 1, 0, 0\rangle + t\,\langle -2, 1, 0\rangle$$
$$l_2 : \langle x, y, z\rangle = \langle 0, 1, 1\rangle + s\,\langle 2. -1, 1\rangle .$$

As l_1 and l_2 are skew, they are contained in two parallel planes. A normal to these 2 parallel planes:

$\vec{n} = \langle -2, 1, 0\rangle \times \langle 2, -1, 1\rangle = \hat{i} + 2\hat{j} = \langle 1, 2, 0\rangle$.

Let $s = 0$ in l_2, we get the pnt. $(0, 1, 1)$ on l_2.

So, an eqn. of the plane containing l_2 is

$$1.(x-0) + 2(y-1) + 0(z-1) = 0$$
$$x + 2y - 2 = 0$$

Let $t = 0$ in l_1, we get $(1, 0, 0)$ on l_1.

$$\text{dist.} = \frac{|1 + 2.0 - 2|}{\sqrt{1^2+2^2}} = \frac{1}{\sqrt{5}}$$

3. (10 points) A particle travels a total distance of 26π along the parametric curve

$$\mathbf{r}(t) = (5\sin t)\mathbf{i} + (5\cos t)\mathbf{j} + 12t\mathbf{k}$$

from the starting point $(0, -5, 12\pi)$. Find its new location.

Notice when $t = \pi$, we get the pnt. $(0, -5, 12\pi)$.

$$26\pi = \int_{\pi}^{b} |\vec{r}\,'(t)|\,dt.$$

$$\vec{r}\,'(t) = \langle 5\cos t, -5\sin t, 12 \rangle$$

$$|\vec{r}\,'(t)| = \sqrt{25\cos^2 t + 25\sin^2 t + 144} = 13$$

$$26\pi = \int_{\pi}^{b} 13\,dt = 13t \Big|_{t=\pi}^{t=b} = 13(b - \pi)$$

$$2\pi = b - \pi$$

$$3\pi = b.$$

When $t = 3\pi$, $\vec{r}(3\pi) = 5\sin(3\pi)\hat{i} + 5\cos(3\pi)\hat{j} + 12(3\pi)\hat{k}$

$$\vec{r}(3\pi) = 0\hat{i} - 5\hat{j} + 36\pi\hat{k}$$

New location $= (0, -5, 36\pi)$

4. (10 points) Evaluate the limit

$$\lim_{(x,y)\to(0,0)} \frac{x^2y^3}{x^4+y^6}$$

or show that it does not exist.

5. (12 points) Find the linearization of the function

$$f(x, y) = \sqrt{12 - x^2 - 7y^2}$$

at $(x, y) = (2, 1)$, and then use it to find the approximate value of $f(2.05, 0.98)$.

$$f_x(x,y) = \frac{-2x}{2\sqrt{12-x^2-7y^2}} = \frac{-x}{\sqrt{12-x^2-7y^2}}$$

$$f_y(x,y) = \frac{-14y}{2\sqrt{12-x^2-7y^2}} = \frac{-7y}{\sqrt{12-x^2-7y^2}}$$

$$f_x(2,1) = \frac{-2}{\sqrt{12-4-7}} = -2$$

$$f_y(2,1) = \frac{-7}{\sqrt{12-4-7}} = -7$$

$$f(2,1) = \sqrt{12-4-7} = 1$$

$$L(x,y) = 1 + (-2)(x-2) + (-7)(y-1)$$

$$L(2.05, 0.98) = 1 - 2\cdot(0.05) - 7(-0.02)$$

$$L(2.05, 0.98) = 1.04 \approx f(2.05, 0.98)$$

6. (16 points) Find the critical points of the function

$$f(x, y) = x^2y - 8y^2 - x^2$$

and classify each of them as a local minimum, local maximum, or saddle point.

$f_x = 2xy - 2x = 2x(y-1) \implies x=0$ or $y=1$.

$f_y = x^2 - 16y$

$\boxed{x=0: \quad 0^2 - 16y = 0, \quad y = 0, \quad (0,0)}$

$\boxed{y=1: \quad x^2 - 16\cdot 1 = 0, \quad x = \pm 4, \quad (-4,1), \ (+4,1)}$

$f_{xx} = 2y - 2$ $\qquad f_{xy} = 2x$

$f_{yy} = -16$ $\qquad D = -16(2y-2) - (2x)^2$

At $(0,0)$, $D(0,0) = -16(-2) - 0^2 = 32 > 0$

$f_{xx}(0,0) = -2 < 0$ local max.

At $(-4,1)$, $D(-4,1) = -16(2-2) - (-8)^2 = -64 < 0$.

saddle point.

At $(4,1)$, $D(4,1) = -16(2-2) - 64 = -64 < 0$

saddle pnt.

7. (16 points) Calculate the integral $\iint_R y\,dA$ where R is the (finite) region lying between the curves $x = y^2$ and $y = x - 2$.

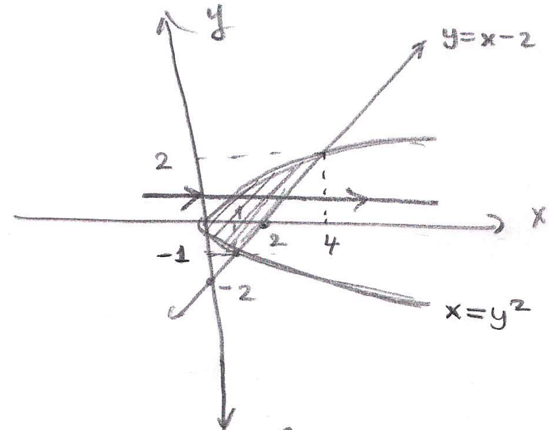
$y + 2 = x$

$y + 2 = y^2$

$0 = y^2 - y - 2$

$0 = (y-2)(y+1)$

$\boxed{\begin{matrix} y=2 \\ x=4 \end{matrix}}$ $\boxed{\begin{matrix} y=-1 \\ x=1 \end{matrix}}$



$$\iint_R y\,dA = \int_{-1}^{2}\int_{y^2}^{y+2} y\,dx\,dy = \int_{-1}^{2}\left[yx\right]\Big|_{x=y^2}^{x=y+2} dy = \int_{-1}^{2}\left(y(y+2) - y\cdot y^2\right)dy$$

$$= \int_{-1}^{2}(y^2+2y-y^3)\,dy = \frac{y^3}{3} + y^2 - \frac{y^4}{4}\Bigg|_{y=-1}^{y=2} =$$

$$= \frac{8}{3} + 4 - \frac{16}{4} - \left[\frac{-1}{3} + 1 - \frac{1}{4}\right] =$$

$$= \frac{9}{3} + 3 - 4 + \frac{1}{4} = 2 + \frac{1}{4} = \frac{9}{4}$$

3

8. (10 points) Circle the correct response for the following questions (no work is required, and there is no partial credit):

(a) The result obtained by reversing the order of integration in the iterated double integral

$$\int_0^1 \int_x^2 x\, dy\, dx$$

is

(i) $\int_0^1 \int_x^2 x\, dx\, dy$

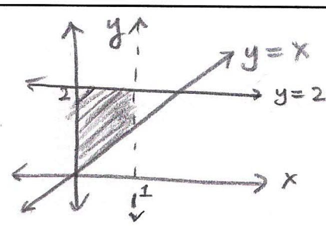
(ii) $\int_0^2 \int_0^y x\, dx\, dy$

(iii) $\int_x^2 \int_0^1 x\, dx\, dy$

(iv) $\int_0^1 \int_y^2 x\, dx\, dy + \int_1^2 \int_0^y x\, dx\, dy$

(v) $\int_0^1 \int_0^y x\, dx\, dy + \int_1^2 \int_0^1 x\, dx\, dy$

(vi) $\int_0^1 \int_0^2 x\, dx\, dy + \int_1^2 \int_y^2 x\, dx\, dy$



(b) In polar coordinates, the integral

$$\int_{-2}^0 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} x\, \underbrace{dy\, dx}_{r\,dr\,d\theta}$$

$x \to r.\cos\theta$

is

(i) $\int_\pi^{2\pi} \int_0^2 r^2 \cos\theta\, dr\, d\theta$

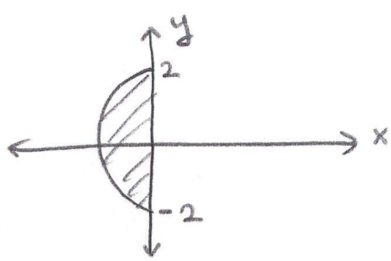
(ii) $\int_{\pi/2}^{3\pi/2} \int_0^2 r^2 \cos\theta\, dr\, d\theta$

(iii) $\int_0^{2\pi} \int_0^2 r^2 \cos\theta\, dr\, d\theta$

(iv) $\int_{-\pi/2}^{\pi/2} \int_0^2 r \cos\theta\, dr\, d\theta$

(v) $\int_{\pi/2}^{3\pi/2} \int_0^2 r \cos\theta\, dr\, d\theta$

(vi) $\int_0^{2\pi} \int_0^2 r \cos\theta\, dr\, d\theta$



Part B

9. (14 points) Evaluate the triple integral $\iiint_E x\, dV$, where E is the solid tetrahedron bounded by the four planes $x = 0$, $y = 0$, $z = 0$ and $x+y+z = 1$, whose vertices are $(0,0,0)$, $(1,0,0)$, $(0,1,0)$, and $(0,0,1)$.

$$\iiint_E x\, dV = \int_0^1 \int_0^{1-x} \int_0^{1-x-y} x\, dz\, dy\, dx = \int_0^1 \int_0^{1-x} x(1-x-y)\, dy\, dx =$$

$$= \int_0^1 \int_0^{1-x} (x - x^2 - xy)\, dy\, dx = \int_0^1 \left(xy - x^2 y - \frac{xy^2}{2} \Big|_{y=0}^{y=1-x} \right) dx =$$

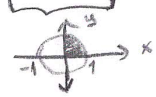
$$= \int_0^1 \left[x(1-x) - x^2 \cdot (1-x) - x \cdot \frac{(1-x)^2}{2} \right] dx = \int_0^1 \left[x - x^2 - x^2 + x^3 - \frac{x}{2} \cdot (1 + x^2 - 2x) \right] dx$$

$$= \int_0^1 \left[x - 2x^2 + x^3 - \frac{x}{2} - \frac{x^3}{2} + x^2 \right] dx = \int_0^1 \left(\frac{x}{2} - x^2 + \frac{x^3}{2} \right) dx =$$

$$= \frac{x^2}{4} - \frac{x^3}{3} + \frac{x^4}{8} \Big|_{x=0}^{x=1} = \frac{1}{4} - \frac{1}{3} + \frac{1}{8} = \frac{1}{24}.$$

10. (14 points) Evaluate the integral $x^2+y^2+z^2=\rho^2$

$$I = \int_0^1 \int_0^{\sqrt{1-x^2}} \int_{-\sqrt{1-x^2-y^2}}^{\sqrt{1-x^2-y^2}} \sqrt{x^2+y^2+z^2}\, \underbrace{dz\, dy\, dx}_{\rho^2 \sin\phi \, d\rho\, d\theta\, d\phi}.$$



$$I = \int_0^{\pi}\int_0^{\pi/2}\int_0^{1} \sqrt{\rho^2}\, \rho^2 \sin\phi \, d\rho\, d\theta\, d\phi = \int_0^{\pi}\int_0^{\pi/2}\int_0^{1} \rho^3 \sin\phi \, d\rho\, d\theta\, d\phi =$$

$$= \int_0^{\pi}\int_0^{\pi/2}\left(\frac{\rho^4}{4}\sin\phi \Big|_{\rho=0}^{\rho=1} \right) d\theta\, d\phi = \int_0^{\pi}\int_0^{\pi/2} \frac{1}{4}\sin\phi \, d\theta\, d\phi$$

$$= \int_0^{\pi} \frac{\pi}{2}\cdot\frac{1}{4}\cdot\sin\phi \, d\phi = \frac{\pi}{8}\cdot(-\cos\phi)\Big|_{\phi=0}^{\phi=\pi} = -\frac{\pi}{8}(-1-1) =$$

$$= \frac{2\pi}{8} = \frac{\pi}{4}$$

11. (12 points) Evaluate the line integral $\int_C (x - y + z + 2)\, ds$, where C is the straight line segment from $(0, 1, 1)$ to $(1, 0, 1)$.

$\vec{v} = (1,0,1) - (0,1,1) = \langle 1, -1, 0 \rangle$

A param. for C: $x(t) = 0 + t$

$y(t) = 1 - t$

$z(t) = 1$

when $t=0$, we have $(0,1,1)$

when $t=1$, " $(1,0,1)$

$\vec{r}(t) = \langle t, 1-t, 1 \rangle$

$\vec{r}\,'(t) = \langle 1, -1, 0 \rangle$

$$\int_C (x-y+z+2)\, ds = \int_0^1 [t-(1-t)+1+2] \cdot \sqrt{1^2+(-1)^2+0^2}\, dt =$$

$$\int_0^1 [t-1+t+3]\, \sqrt{2}\, dt = \sqrt{2} \int_0^1 (2t+2)\, dt = \sqrt{2}\, (t^2+2t) \Big|_{t=0}^{t=1}$$

$$= \sqrt{2}\, (1+2) = 3\sqrt{2}.$$

12. (12 points) Let f be a scalar function and $\mathbf{F}$ be a vector field. For each expression, identify whether it is a scalar function, a vector field, or nonsense by circling the appropriate response (no work is required, and there is no partial credit).
Note that $\text{grad}(f) = \nabla f$ denotes the gradient of f.

• $\text{curl}(f)$	**Scalar function**	**Vector field**	**Nonsense**
• $\text{grad}(f)$	**Scalar function**	**Vector field**	**Nonsense**
• $\text{div}(\mathbf{F})$	**Scalar function**	**Vector field**	**Nonsense**
• $\text{curl}(\text{grad}(f))$	**Scalar function**	**Vector field**	**Nonsense**
• $\text{grad}(\mathbf{F})$	**Scalar function**	**Vector field**	**Nonsense**
• $\text{grad}(\text{div}(\mathbf{F}))$	**Scalar function**	**Vector field**	**Nonsense**
• $\text{div}(\text{grad}(f))$	**Scalar function**	**Vector field**	**Nonsense**
• $\text{grad}(\text{div}(f))$	**Scalar function**	**Vector field**	**Nonsense**
• $\text{curl}(\text{curl}(\mathbf{F}))$	**Scalar function**	**Vector field**	**Nonsense**
• $\text{div}(\text{div}(\mathbf{F}))$	**Scalar function**	**Vector field**	**Nonsense**
• $(\text{grad}(f)) \times (\text{div}(\mathbf{F}))$	**Scalar function**	**Vector field**	**Nonsense**
• $\text{div}(\text{curl}(\text{grad}(f))))$	**Scalar function**	**Vector field**	**Nonsense**

13. (12 points) Find all possible values of the constants a and b such that the vector field

$$\mathbf{F}(x,y,z) = (2bxz^3 + ayz + 2xy)\mathbf{i} + (2by + ax^2 + axz)\mathbf{j} + (axy + bz + bx^2z^2)\mathbf{k}$$

is conservative (i.e., that the work done by the field on a particle moving through space does not depend on the particle's path).

Since $\vec{F}$ is conservative, there exists f such that $\vec{\nabla f} = \vec{F}$:

$f_x(x,y,z) = 2bxz^3 + ayz + 2xy$

$f_y(x,y,z) = 2by + ax^2 + axz$

$f_z(x,y,z) = axy + bz + bx^2z^2$

$$\left.\begin{aligned} f_{xy}(x,y,z) &= az + 2x \\ f_{yx}(x,y,z) &= 2ax + az \end{aligned}\right\} \Rightarrow \boxed{\begin{aligned} 2a &= 2 \\ a &= 1 \end{aligned}}$$

$$\left.\begin{aligned} f_{xz}(x,y,z) &= 6bxz^2 + ay \\ f_{zx}(x,y,z) &= ay + 2bxz^2 \end{aligned}\right\} \Longrightarrow \begin{aligned} 6bxz^2 &= 2bxz^2 \\ 2b &= 6b \\ &\boxed{b = 0} \end{aligned}$$

$$\left.\begin{aligned} f_{yz}(x,y,z) &= ax \\ f_{zy}(x,y,z) &= ax \end{aligned}\right\} \checkmark$$

14. (12 points) Compute the work integral $\oint_C \mathbf{F}\cdot d\mathbf{r}$, where

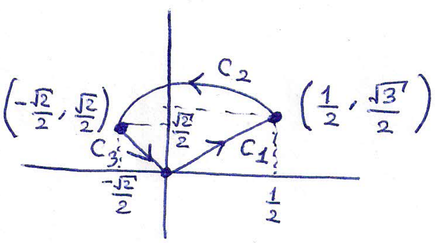
$$\mathbf{F}(x,y) = \left(x^{2015} + x^2 y\right)\mathbf{i} + \left(xy^2 + 2e^y\right)\mathbf{j}$$

and the closed path C, oriented counterclockwise, consists of the following three pieces:

C_1 : the line segment from $(0,0)$ to $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$,

C_2 : the curve $y = \sqrt{1-x^2}$ with $-\frac{\sqrt{2}}{2} \le x \le \frac{1}{2}$,

C_3 : the line segment from $\left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$ to $(0,0)$.



C_1:	$x = 0 + \frac{1}{2}t$ $y = 0 + \frac{\sqrt{3}}{2}t$	$0 \le t \le 1$
C_2:	$x = -t$ $y = \sqrt{1-t^2}$	$-\frac{1}{2} \le t \le \frac{\sqrt{2}}{2}$
C_3:	$x = -\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}t$ $y = \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}t$	$0 \le t \le 1$.

$$\int_{C_1} \vec{F}\cdot d\vec{r} = \int_0^1 \left\langle \left(\frac{t}{2}\right)^{2015} + \left(\frac{t}{2}\right)^2 \cdot \frac{\sqrt{3}}{2}t \,,\; \frac{t}{2}\left(\frac{\sqrt{3}}{2}t\right)^2 + 2\cdot e^{\frac{\sqrt{3}}{2}t} \right\rangle \cdot \left\langle \frac{1}{2}, \frac{\sqrt{3}}{2} \right\rangle dt$$

$$= \int_0^1 \left(\frac{1}{2}\cdot\frac{t^{2015}}{2^{2015}} + \frac{\sqrt{3}}{4}\cdot\frac{t^3}{4} + \frac{\sqrt{3}}{4}\cdot t\cdot\frac{3}{4}\cdot t^2 + \sqrt{3}\cdot e^{\frac{\sqrt{3}}{2}t} \right) dt$$

$$= \frac{1}{2^{2016}}\cdot\frac{t^{2016}}{2016} + \frac{\sqrt{3}}{16}\cdot\frac{t^4}{4} + \frac{3\sqrt{3}}{16}\cdot\frac{t^4}{4} + 2\cdot e^{\frac{\sqrt{3}}{2}t} \Bigg|_{t=0}^{t=1}$$

$$= \frac{1}{2^{2016}\cdot 2016} + \frac{\sqrt{3}}{64} + \frac{3\sqrt{3}}{64} + 2\cdot e^{\sqrt{3}/2} - 2\cdot e^0$$

$$= \frac{1}{2^{2016}\cdot 2016} + \frac{\sqrt{3}}{16} + 2\cdot e^{\sqrt{3}/2} - 2 .$$

14 continues:

$$\int_{C_2} \vec{F}\cdot d\vec{r} = \int_{-\frac{1}{2}}^{\sqrt{2}/2} \langle (-t)^{2015} + (-t)^2 \cdot \sqrt{1-t^2},\ (-t)(1-t^2) +$$

$$+ 2 \cdot e^{\sqrt{1-t^2}} \rangle \cdot \langle -1, \frac{-t}{\sqrt{1-t^2}} \rangle dt = \int_{-\frac{1}{2}}^{\sqrt{2}/2} t^{2015} - \frac{2t}{\sqrt{1-t^2}} \cdot e^{\sqrt{1-t^2}} dt =$$

$$= \left[\frac{t^{2016}}{2016} + 2 \cdot e^{\sqrt{1-t^2}} \right] \Bigg|_{t=-\frac{1}{2}}^{t=\frac{\sqrt{2}}{2}=\frac{1}{\sqrt{2}}} = \frac{1}{2016 \cdot 2^{1008}} + 2 \cdot e^{1/\sqrt{2}} - \frac{1}{2016 \cdot 2^{2016}} - 2 \cdot e^{\frac{\sqrt{3}}{2}}$$

$$\int_{C_3} \vec{F}\cdot d\vec{r} = \int_0^1 \langle \left(-\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}t\right)^{2015} + \left(-\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}t\right)^2 \cdot \left(\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}t\right), \left(-\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}t\right)$$

$$\left(\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}t\right)^2 + 2 \cdot e^{\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}t} \rangle \cdot \langle \frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2} \rangle dt =$$

$$= \int_0^1 \frac{1}{2^{1008}} \cdot (-1+t)^{2015} - \frac{1}{4}(-1+t)^3 + \frac{(1-t)^3}{4} - \sqrt{2} \cdot e^{\sqrt{2}/2} \cdot e^{-\sqrt{2}/2\, t} dt$$

$$= \left[\frac{1}{2^{1008}} \cdot \frac{(-1+t)^{2016}}{2016} - \frac{2}{4} \frac{(-1+t)^4}{4} + 2 \cdot e^{\sqrt{2}/2} \cdot e^{-\sqrt{2}/2\, t} \right]_{t=0}^{t=1}$$

$$= 2 \cdot e^{\sqrt{2}/2} \cdot e^{-\sqrt{2}/2} - \left[\frac{1}{2016 \cdot 2^{1008}} - \frac{1}{8} + 2 \cdot e^{\sqrt{2}/2} \right] = 2 - \frac{1}{2016 \cdot 2^{1008}} + \frac{1}{8} - 2 \cdot e^{\frac{\sqrt{2}}{2}}$$

$$\int_{C_1} + \int_{C_2} + \int_{C_3} = \frac{\sqrt{3}}{16} + \frac{1}{8}.$$

15. (12 points) Find the surface area of the part of the paraboloid $z = x^2 + y^2$ that lies under the plane $z = 4$.

$$A = \iint_D \sqrt{1+(2x)^2+(2y)^2}\, dA = \iint_D \sqrt{1+4(x^2+y^2)}\, dA =$$

$$x^2+y^2=4$$

$$r=2$$

$$= \int_0^{2\pi}\int_0^{2} \sqrt{1+4r^2}\, r\,dr\,d\theta = \int_0^{2\pi} d\theta \cdot \int_0^{2} r \cdot \sqrt{1+4r^2}\, dr =$$

$$= 2\pi \cdot \frac{1}{8} \cdot \frac{2}{3} \cdot (1+4r^2)^{3/2} \Big|_{r=0}^{r=2} = \frac{\pi}{6} \cdot \left[(17)^{3/2} - 1\right].$$

16. (12 points) Compute the flux of the vector field

$$\mathbf{F}(x,y,z) = \left(-3x^2y\right)\mathbf{i} + (z-y)\,\mathbf{j} + (2x)\,\mathbf{k}$$

through the surface S given by the part of the plane $z = 1+2x+y$ lying above the rectangle $0 \le x \le 2$, $0 \le y \le 3$, with upward orientation.

$$\vec{F}(x,y,z) = (-3x^2y)\hat{\imath} + (z-y)\hat{\jmath} + (2x)\hat{k}$$

$x = u$ $\qquad \vec{r}(u,v) = \langle u, v, 1+2u+v \rangle$.

$y = v$

$z = 1+2u+v$ $\qquad \vec{r}_u = \langle 1, 0, 2 \rangle$

$\vec{r}_v = \langle 0, 1, 1 \rangle$

$\vec{r}_u \times \vec{r}_v = \hat{k} - 2\hat{\imath} - \hat{\jmath} = \langle -2, -1, 1 \rangle$.

$$\text{Flux} = \int_S \vec{F} \cdot d S = \iint_D \vec{F} \cdot (\vec{r}_u \times \vec{r}_v)\, dA =$$

$$= \int_0^3 \int_0^2 (-3u^2v\hat{\imath} + (1+2u)\hat{\jmath} + 2u\hat{k}) \cdot (-2\hat{\imath} - \hat{\jmath} + \hat{k})\, du\, dv$$

$$= \int_0^3 \int_0^2 (6u^2v - 1 - \cancel{2u} + \cancel{2u})\, du\, dv = \int_0^3 \left[\frac{6u^3}{3}v - u\right]_{u=0}^{u=2} dv =$$

$$= \int_0^3 (2 \cdot 2^3 \cdot v - 2)\, dv = \int_0^3 (16v - 2)\, dv = (8v^2 - 2v)\Big|_{v=0}^{v=3} =$$

$$= 72 - 6 = 66$$