1. (9 points) Find all the critical points of the function $f(x, y) = 2y^3 + yx^2 - y - 2024$ and classify them as local maxima, local minima, or saddle points.

$\frac{\partial f}{\partial x} = 2yx = 0 \Rightarrow x = 0$ or $y = 0$

$\frac{\partial f}{\partial y} = 6y^2 + x^2 - 1 = 0 \Rightarrow 6y^2 + x^2 = 1$

If $x = 0 \Rightarrow 6y^2 = 1 \Rightarrow y = \pm\frac{1}{\sqrt{6}} \rightarrow (0, \pm\frac{1}{\sqrt{6}})$

If $y = 0 \Rightarrow x^2 = 1 \Rightarrow x = \pm 1. \Rightarrow (\pm 1, 0)$

$D = f_{xx}f_{yy} - f_{xy}^2 = (2y)(12y) - (2x)^2 = 4(6y^2 - x^2).$

- $D(0, \pm\frac{1}{\sqrt{6}}) = 4(6(\frac{1}{\sqrt{6}})^2 - 0) > 0$. $f_{xx}(0, \pm\frac{1}{\sqrt{6}}) = \pm\frac{2}{\sqrt{6}}$

So, $(0, 1/\sqrt{6})$ is a local min.

$(0, -\frac{1}{\sqrt{6}})$ is a local max.

- $D(\pm 1, 0) = 4(6(0)^2 - 1^2) < 0 \Rightarrow$ Both $(1, 0), (-1, 0)$ are saddle pts.

2. (9 points) Find the maximum and minimum of the function $f(x, y) = x - 2y + 5$ subject to the constraint $x^2 + y^2 - 3xy = 20$.

Let $g(x,y) = x^2 + y^2 - 3xy$. Then,

$$\begin{cases} \nabla f = \lambda \nabla g \\ g = 20 \end{cases} \Rightarrow \begin{cases} \langle 1, -2 \rangle = \lambda \langle 2x - 3y, 2y - 3x \rangle \\ x^2 + y^2 - 3xy = 20 \end{cases}$$

$1 = \lambda(2x - 3y)$ $\rightarrow$ note: If $\lambda = 0$, then $1 = 0$ which is a contradiction.

$1 = \lambda(2x-3y)$ $\to$ note: If $\lambda=0$, then $1=0$ which is a contradiction.

So, $\lambda \neq 0 \Rightarrow \boxed{\frac{1}{\lambda} = 2x-3y.}$

Also have $-2 = \lambda(2y-3x) \Rightarrow \boxed{\frac{1}{\lambda} = -y+\frac{3}{2}x}$

Then, $2x-3y = -y+\frac{3}{2}x$

$\Rightarrow \frac{x}{2} = 2y \Rightarrow \boxed{x=4y}$

Then, the constraint says $(4y)^2+y^2-3(4y)y = 20$

$\Rightarrow 16y^2+y^2-12y^2 = 20$

$\Rightarrow y^2 = 4$

$\Rightarrow \boxed{y=\pm 2}$

$x=4y \Rightarrow$ we get $(x,y)=(8,2)$ w/ $\lambda = \frac{1}{2x-3y} = \frac{1}{10}$

and $(x,y)=(-8,-2)$ w/ $\lambda = -\frac{1}{10}$

$f = x-2y+5 \Rightarrow f(8,2) = 9 \leftarrow$ max

$f(-8,-2) = 1 \leftarrow$ min.

3. (6 points) Let $f(x,y) = x^2+3y^2$.

(a) Find the directional derivative of f in the direction of the vector $\mathbf{v} = \langle 1,-1\rangle$ at the point $(2,1)$.

$\nabla f = \langle 2x, 6y\rangle \Rightarrow \nabla f(2,1) = \langle 4,6\rangle$

Let $\vec{u} = \frac{\vec{v}}{|\vec{v}|} = \frac{1}{\sqrt{2}}\langle 1,-1\rangle$. Then, Let $u = \frac{\vec{v}}{|\vec{v}|} = \frac{1}{\sqrt{2}}\langle 1,$

$$D_{\vec{u}} f(2,1) = \nabla f(2,1) \cdot \vec{u} = \frac{1}{\sqrt{2}}\langle 4,6\rangle \cdot \langle 1,-1\rangle$$

$$= \frac{-2}{\sqrt{2}} = \boxed{-\sqrt{2}}$$

(b) Find the unit vector in the direction for which $f(x, y)$ is increasing fastest at the point $(2, 1)$.

f increases fastest in the direction of ∇f.

$$\rightarrow \quad \frac{\nabla f(2,1)}{|\nabla f(2,1)|} = \frac{\langle 4,6\rangle}{|\langle 4,6\rangle|} = \frac{2\langle 2,3\rangle}{|2\langle 2,3\rangle|} = \frac{\langle 2,3\rangle}{\sqrt{2^2+3^2}}$$

$$= \boxed{\frac{\langle 2,3\rangle}{\sqrt{13}}}$$

4. (8 points) Complete each part below, showing all work.

(a) Evaluate the double integral $\iint_D e^x \cos y \, dA$ where D is the rectangular region

$$D = \{\, (x, y) \in \mathbb{R}^2 \,|\, 1 \le x \le 2, 0 \le y \le \pi/3 \,\}.$$

$$\rightarrow \quad \int_0^{\pi/3} \int_1^2 e^x \cos y \, dx \, dy = \int_0^{\pi/3} \cos y \; e^x \Big|_1^2 \, dy$$

$$= (e^2 - e) \int_0^{\pi/3} \cos y \, dy$$

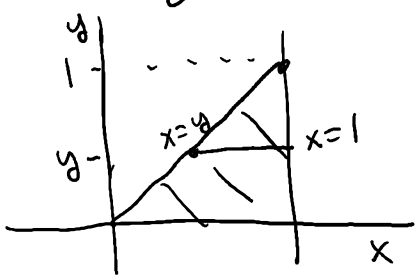
$$= (e^2 - e) \sin y \Big|_0^{\pi/3}$$

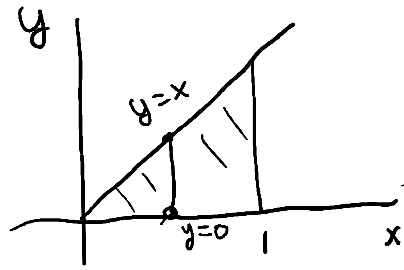
$$= \boxed{(e^2 - e)\frac{\sqrt{3}}{2}}$$

(b) Evaluate the iterated integral $\displaystyle\int_0^1 \int_y^1 e^{x^2} \, dx \, dy$.

$$D = \{(x,y) \mid 0 \le y \le 1, \; y \le x \le 1\}$$

$D = \{(x,y) \mid 0 \le y \le 1,\ y \le x \le 1\}$





$\Rightarrow D = \{(x,y) \mid 0 \le x \le 1,\ 0 \le y \le x\}$

$$\Rightarrow \int_0^1 \int_y^1 e^{x^2}\,dx\,dy = \iint_D e^{x^2}\,dA$$

$$= \int_{x=0}^{1} \int_{y=0}^{x} e^{x^2}\,dy\,dx$$

$$= \int_0^1 e^{x^2}\,x\,dx$$

$u = x^2$

$\frac{1}{2}du = dx$

$u(0) = 0$

$u(1) = 1$

$$= \frac{1}{2}\int_{u=0}^{1} e^u\,du$$

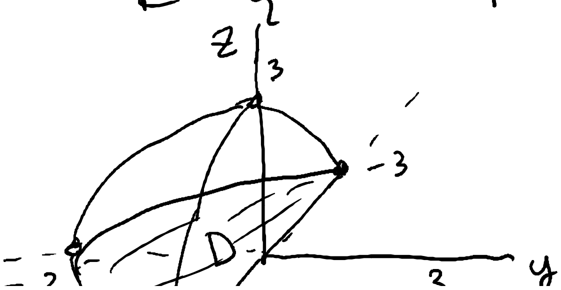
$$= \frac{1}{2}e^u\Big|_0^1 = \boxed{\frac{e-1}{2}}$$

5. (9 points) Evaluate

$$\int_{-3}^{0}\int_{-\sqrt{9-y^2}}^{\sqrt{9-y^2}}\int_0^{\sqrt{9-x^2-y^2}} x^2z + y^2z + z^3\ dz\ dx\ dy$$

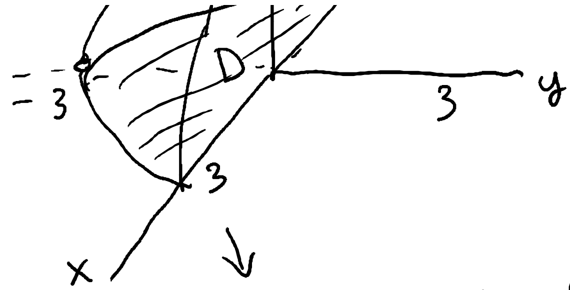
by changing to spherical coordinates.

$\rightarrow \quad E = \{(x,y,z) \mid -3 \le y \le 0,\ -\sqrt{9-y^2} \le x \le \sqrt{9-y^2},\ 0 \le z \le \sqrt{9-x^2-y^2}\}$



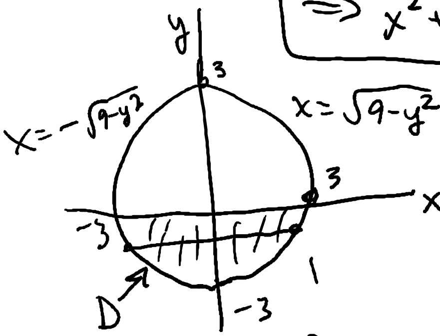
Note: $z = \sqrt{9-x^2-y^2}$ is upper hemi-sphere since $z^2 = 9 - x^2 - y^2$

$\Rightarrow x^2 + y^2 + z^2 = 9$



since $z = \sqrt{9-x^2-y^2}$

$\Rightarrow x^2+y^2+z^2=9$



E = quarter of solid sphere lying above D shown.

since $-3 \le y \le 0$

$$\Rightarrow E = \{(\rho, \theta, \phi) \mid 0 \le \rho \le 3,\ \pi \le \theta \le 2\pi,\ 0 \le \phi \le \tfrac{\pi}{2}\}$$

$$dV = \rho^2 \sin\phi \, d\phi, \quad \begin{cases} x = \rho \cos\theta \sin\phi \\ y = \rho \sin\theta \sin\phi \\ z = \rho \cos\phi \end{cases}$$

$$\iiint_E (x^2 z + y^2 z + z^3)\, dV = \iiint_E z(x^2+y^2+z^2)\, dV$$

$$= \int_{\rho=0}^{3} \int_{\theta=\pi}^{2\pi} \int_{\phi=0}^{\pi/2} \rho\cos\phi\,(\rho^2)\,\rho^2 \sin\phi \, d\phi\, d\theta\, d\rho$$

$$= \int_0^3 \rho^5\, d\rho \cdot \int_\pi^{2\pi} d\theta \cdot \int_0^{\pi/2} \sin\phi\cos\phi\, d\phi$$

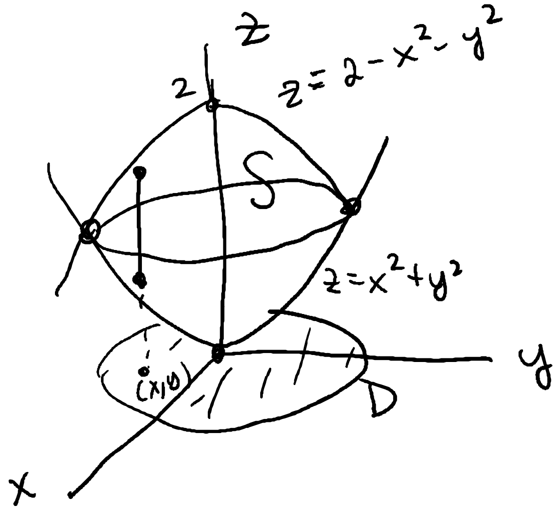
$$= \frac{1}{6}(3^6) \cdot \pi \cdot \int_0^{\pi/2} \frac{\sin(2\phi)}{2}\, d\phi$$

$$= \frac{3^6}{6} \cdot \pi \cdot \left(-\frac{\cos(2\phi)}{4}\right)\Big|_0^{\pi/2}$$

$$= \frac{3^6}{6} \cdot \pi \cdot \frac{1}{2} = \boxed{\frac{3^5}{4}\pi}$$

6. (9 points) Let S be the solid bounded by the surfaces $z = x^2 + y^2$ and $z = 2 - x^2 - y^2$.

(a) Set up (but do NOT evaluate) an iterated integral in Cartesian coordinates to find the volume of S.

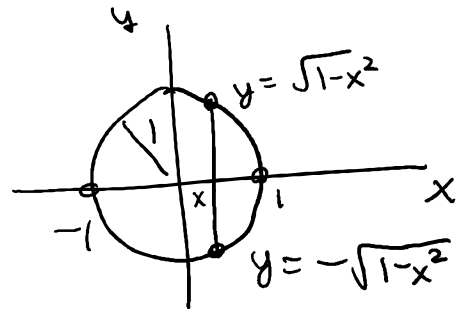


D = region enclosed by the intersection

$2 - x^2 - y^2 = x^2 + y^2$

$\Rightarrow 2(x^2+y^2) = 2$

$\Rightarrow x^2 + y^2 = 1$



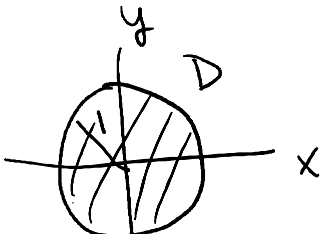
$$S = \{(x,y,z) \mid (x,y) \text{ in } D \text{ and } x^2+y^2 \le z \le 2 - x^2 - y^2\}$$

$$= \{(x,y,z) \mid -1 \le x \le 1,\ -\sqrt{1-x^2} \le y \le \sqrt{1-x^2},\ x^2+y^2 \le z \le 2-x^2-y^2\}$$

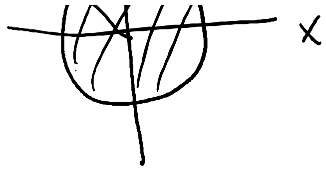
$$\Rightarrow \text{Vol}(S) = \iiint_S dV = \iint_{(x,y)\in D} \left[\int_{z=x^2+y^2}^{2-x^2-y^2} dz\right] dA$$

$$= \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_{z=x^2+y^2}^{2-x^2-y^2} dz\, dy\, dx$$

(b) Set up (but do NOT evaluate) an iterated integral in cylindrical coordinates to find the volume of S.



In polar $D = \{(r,\theta) \mid 0 \le r \le 1,\ 0 \le \theta \le 2\pi\}$



$$\text{vol}(S) = \iiint_S dV = \iint_{(x,y)\in D} \left[\int_{z=x^2+y^2}^{2-x^2-y^2} dz \right] dA$$

$$= \boxed{\int_{\theta=0}^{2\pi} \int_{r=0}^{1} \int_{z=r^2}^{2-r^2} dz \, r \, dr \, d\theta}$$

(c) Evaluate either integral in parts (a) or (b) to find the volume of S.

$$\text{vol}(S) = \int_{\theta=0}^{2\pi} \int_{r=0}^{1} \int_{z=r^2}^{2-r^2} dz \, r \, dr \, d\theta$$

$$= \int_0^{2\pi} \int_{r=0}^{1} (2 - r^2 - r^2) \, r \, dr \, d\theta$$

$$= 2 \int_0^{2\pi} d\theta \int_0^1 (r - r^3) \, dr$$

$$= 4\pi \left(\tfrac{1}{2} r^2 - \tfrac{1}{4} r^4 \right) \Big|_0^1 = 4\pi \left(\tfrac{1}{2} - \tfrac{1}{4} \right) = \pi$$