1. (6 points) Consider the surface

$$e^{xz} = \frac{y}{z} + \frac{x}{y}.$$

Viewing z as an implicitly defined function of x, y, answer the following.

(a) Find $\frac{\partial z}{\partial x}$. (Do not simplify.)

$$\frac{\partial}{\partial x}\left(e^{xz}\right) = \frac{\partial}{\partial x}\left(\frac{y}{z} + \frac{x}{y}\right)$$

$$\Rightarrow \quad z e^{xz} + x\frac{\partial z}{\partial x} e^{xz} = -\frac{y}{z^2}\frac{\partial z}{\partial x} + \frac{1}{y}$$

$$\Rightarrow \quad \frac{\partial z}{\partial x}\left(\frac{y}{z^2} + x e^{xz}\right) = \frac{1}{y} - z e^{xz}$$

$$\Rightarrow \quad \boxed{\frac{\partial z}{\partial x} = \frac{\frac{1}{y} - z e^{xz}}{\frac{y}{z^2} + x e^{xz}}}$$

(b) Find $\frac{\partial z}{\partial y}$. (Do not simplify.)

$$\frac{\partial}{\partial y}\left(e^{xz}\right) = \frac{\partial}{\partial y}\left(\frac{y}{z} + \frac{x}{y}\right)$$

$$\Rightarrow \quad x\frac{\partial z}{\partial y} e^{xz} = \frac{1}{z} - \frac{y}{z^2}\frac{\partial z}{\partial y} - \frac{x}{y^2}$$

$$\Rightarrow \quad \frac{\partial z}{\partial y}\left(x e^{xz} + \frac{y}{z^2}\right) = \frac{1}{z} - \frac{x}{y^2}$$

$$\Rightarrow \quad \boxed{\frac{\partial z}{\partial y} = \frac{\frac{1}{z} - \frac{x}{y^2}}{x e^{xz} + \frac{y}{z^2}}}$$

2. (6 points) Let $P = (0, -1, 3)$, $Q = (2, 0, 1)$, and $R = (1, -1, 1)$.

(a) Find the area of the triangle with vertices P, Q, and R.

$$\vec{PQ} = \langle 2-0, 0-(-1), 1-3 \rangle = \langle 2, 1, -2 \rangle$$

$$\vec{PR} = \langle 1-0, -1-(-1), 1-3 \rangle = \langle 1, 0, -2 \rangle$$

$$\vec{PQ} \times \vec{PR} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 1 & -2 \\ 1 & 0 & -2 \end{vmatrix} = \left\langle \begin{vmatrix} 1 & -2 \\ 0 & -2 \end{vmatrix}, -\begin{vmatrix} 2 & -2 \\ 1 & -2 \end{vmatrix}, \begin{vmatrix} 2 & 1 \\ 1 & 0 \end{vmatrix} \right\rangle$$

$$= \langle -2, -(-4+2), -1 \rangle$$

$$= \langle -2, 2, -1 \rangle$$

$$\text{area } \Delta PQR = \frac{1}{2} \left| \vec{PQ} \times \vec{PR} \right| = \frac{1}{2} \sqrt{(-2)^2 + 2^2 + (-1)^2} = 3/2$$

(b) Find an equation of the plane that contains P, Q, and R.

Use $\vec{n} = \langle -2, 2, -1 \rangle$ and $(x_0, y_0, z_0) = \langle 0, -1, 3 \rangle$

w/ $\vec{n} \cdot \langle x - x_0, y - y_0, z - z_0 \rangle = 0$

$\Rightarrow \quad -2x + 2(y+1) - (z-3) = 0$

$\Rightarrow \quad \boxed{-2x + 2y - z = -5}$

3. (6 points) Suppose that an airline company has two airplanes and assume that the location of each plane at time $t \geq 0$ is described by the curves $\vec{r_1}(t) = < 4t, 2t^2, t >$ and $\vec{r_2}(t) = < 2 + 5t, 4 + 7t, 2t - 1 >$. Answer the following and justify your reasoning.

(a) Do these airplanes crash into each other?

No : Crash at time t requries $t = 2t - 1$

$\Rightarrow \boxed{t = 1}$

$\Rightarrow \boxed{t=1}$

But $\vec{r}_1(1) = \langle 4, 2, 1 \rangle \neq \langle 7, 11, 1 \rangle = \vec{r}_2(1)$

(b) Do these airplanes pass through a common point, that is, do their paths intersect?

Yes: $\begin{cases} 4t = 2+5s \\ 2t^2 = 4+7s \\ t = 2s-1 \end{cases} \Rightarrow 4(2s-1) = 2+5s$

$\Rightarrow 8s - 4 = 2 + 5s$

$\Rightarrow 3s = 6$

$\Rightarrow \underline{s=2}$

Then, $\underline{t = 2s-1 = 3}$

In particular, $\vec{r}_1(3) = \langle 4(3), 2(3)^2, 3 \rangle$
$= \langle 12, 18, 3 \rangle$

and $\vec{r}_2(2) = \langle 2+5(2), 4+7(2), 2(2)-1 \rangle$
$= \langle 12, 18, 3 \rangle$

So, both planes pass through $(12, 18, 3)$.

4. (9 points) Consider the curve $\vec{r}(t) = \langle 8-4t, 5, 3t-2 \rangle$.

(a) Find the arc length function $s(t)$ for the curve measured from the point $P(4,5,1)$ in the direction of increasing t.

$\vec{r}(1) = \langle 4, 5, 1 \rangle$ so $s(t) = \int_1^t |\vec{r}\,'(u)|\,du$, where

$\vec{r}\,'(t) = \langle -4, 5, 3 \rangle \Rightarrow |\vec{r}\,'(t)| = \sqrt{(-4)^2 + 5^2 + 3^2}$
$= \sqrt{50} = 5\sqrt{2}$

Then, $s(t) = \int_1^t 5\sqrt{2}\, du = \underline{5\sqrt{2}(t-1)}$

(b) Reparametrize the curve with respect to arc length s starting from P.

$s = 5\sqrt{2}(t-1) \Rightarrow \frac{s}{5\sqrt{2}} = t-1 \Rightarrow t = 1 + \frac{s}{5\sqrt{2}}$

Then, $\vec{r} = \langle 8-4t, 5, 3t-2 \rangle$

$= \langle 8 - 4(1+\frac{s}{5\sqrt{2}}), 5, 3(1+\frac{s}{5\sqrt{2}}) - 2 \rangle$

$\Rightarrow \underline{\vec{r} = \langle 4 - \frac{4}{5\sqrt{2}}s, 5, 1 + \frac{3}{5\sqrt{2}}s \rangle}$

(c) Find the point 15 units along the curve in the direction of increasing t from P.

$\underline{s=15} \Rightarrow \vec{r} = \langle 4 - \frac{4}{5\sqrt{2}}(15), 5, 1 + \frac{3}{5\sqrt{2}}(15) \rangle$

$\downarrow$

$\underline{\text{point} = \left(4 - \frac{12}{\sqrt{2}}, 5, 1 + \frac{9}{\sqrt{2}}\right).}$

5. (12 points) A particle has acceleration $\vec{a}(t) = \langle -\sin t, \sin t, \cos t \rangle$. Suppose its initial velocity is $\vec{v}(0) = \langle 1, 0, 0 \rangle$ and initial position is $\vec{r}(0) = \langle 0, 0, 0 \rangle$.

(a) Find its velocity $\vec{v}(t)$ at time t. Hint: recall $\vec{a}(t) = \frac{d\vec{v}}{dt}$.

$\vec{v}(t) = \int \vec{a}(t)\,dt = \langle \cos t + C_1, -\cos t + C_2, \sin t + C_3 \rangle$

$\vec{v}(0) = \langle 1, 0, 0 \rangle \Rightarrow \cos(0) + C_1 = 1 \Rightarrow \boxed{C_1 = 0}$

$\vec{v}(0) = \langle 1, 0, 0 \rangle \Rightarrow$ $\cos(0) + C_1 = 1 \Rightarrow \boxed{C_1 = 0}$

$-\cos(0) + C_2 = 0 \Rightarrow \boxed{C_2 = 1}$

$\sin(0) + C_3 = 0 \Rightarrow \boxed{C_3 = 0}$

$\Rightarrow \underline{\vec{v}(t) = \langle \cos t, -\cos t + 1, \sin t \rangle}$

(b) Find its position $\vec{r}(t)$ at time t.

$\vec{r}(t) = \int \vec{v}(t)\,dt = \langle \sin t + E_1, -\sin t + t + E_2, -\cos t + E_3 \rangle$

$\vec{r}(0) = \langle 0, 0, 0 \rangle \Rightarrow \langle E_1, E_2, -1 + E_3 \rangle = \langle 0, 0, 0 \rangle$

$\Rightarrow E_1 = E_2 = 0, \ E_3 = 1$

$\Rightarrow \underline{\vec{r}(t) = \langle \sin t, -\sin t + t, -\cos t + 1 \rangle}$

(c) Find the unit tangent vector $\mathbf{T}$ of the position function $\vec{r}$ at $t = \pi$.

$\vec{v}(\pi) = \langle \cos\pi, -\cos\pi + 1, \sin\pi \rangle = \langle -1, 2, 0 \rangle$

$$\Rightarrow \hat{T}(\pi) = \frac{\vec{v}(\pi)}{|\vec{v}(\pi)|} = \frac{\langle -1, 2, 0 \rangle}{\sqrt{(-1)^2 + 2^2}} = \frac{1}{\sqrt{5}}\langle -1, 2, 0 \rangle$$

(d) Find a vector equation for the tangent line to the curve defined by $\vec{r}$ at $t = \pi$.

- Point <u>on</u> line $= \vec{r}(\pi) = \langle \sin\pi, -\sin\pi + \pi, -\cos\pi + 1 \rangle$
  $= \langle 0, \pi, 2 \rangle$

∘ tangent vec. for line $= \vec{v}(\pi) = \langle -1, 2, 0 \rangle$.

So, line is $\vec{l}(t) = \vec{r}(\pi) + \vec{v}(\pi)t$

$= \langle 0, \pi, 2 \rangle + t\langle -1, 2, 0 \rangle$

$= \underline{\langle -t, \pi + 2t, 2 \rangle}$

6. (5 points) Let $f(x, y)$ be given by

$$f(x, y) = \sin(\pi xy) + y^2 e^x + 2^6.$$

Find the equation of the tangent plane to the surface $z = f(x, y)$ at the point $(1, 0)$.

$f_x = \pi y \cos(\pi xy) + y^2 e^x \rightarrow f_x(1,0) = 0$

$f_y = \pi x \cos(\pi xy) + 2y e^x \Rightarrow f_y(1,0) = \pi$

$f(1,0) = \sin(0) + 0^2 e + 2^6 = 2^6$

Then, $z = f(1,0) + f_x(1,0)(x-1) + f_y(1,0)(y-0)$

$\rightarrow \boxed{z = 2^6 + \pi y}$

7. (6 points) For each of the given limits, either evaluate it or show that it does not exist:

(a) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2 + 2xy - 3y^2 + 1}{(e^{xy} + 3)^{\frac{1}{3}}}$

$= \dfrac{0^2 + 2(0)(0) - 3(0)^2 + 1}{(e^0 + 3)^{1/3}}$

$= \dfrac{1}{4^{1/3}}$

$$= \frac{1}{4^{1/3}}$$

(b) $\lim_{(x,y)\to(0,0)} \frac{y\sin x}{x^2+y^2}$

$\to$ let $f(x,y) = \frac{y \sin x}{x^2+y^2}$.

Along y-axis: $f(0,y) = \frac{y\sin(0)}{0^2+y^2} = \frac{0}{y^2} = 0$.

Then, $\lim_{y\to 0} f(0,y) = 0$ so $f \to 0$ as $(x,y)\to(0,0)$

along the y-axis.

Along $y=x$: $f(x,x) = \frac{x\sin x}{x^2+x^2} = \frac{\sin x}{2x}$

Then, $\lim_{x\to 0} f(x,x) = \lim_{x\to 0} \frac{\sin x}{2x} \overset{0/0}{=} \lim_{x\to 0} \frac{\cos x}{2} = \frac{1}{2}$

(used L'Hospital's rule)

So, $f \to \frac{1}{2}$ as $(x,y)\to(0,0)$ along $y=x$.

Since $\frac{1}{2} \neq 0$, we know $\lim_{(x,y)\to(0,0)} \frac{y\sin x}{x^2+y^2}$ does not exist.