

# Math 164: Multidimensional Calculus

Final Exam

December 15, 2015

NAME (please print legibly): Solutions

Your University ID Number: \_\_\_\_\_

Indicate your instructor with a check in the appropriate box:

Bobkova	TR 12:30-1:45pm	<input type="checkbox"/>
Chen	MW 3:25-4:40pm	<input type="checkbox"/>
Dummit	TR 3:25-4:40pm	<input type="checkbox"/>
Salur	MWF 09:00-09:50am	<input type="checkbox"/>

- You are responsible for checking that this exam has all 19 pages.
- No calculators, phones, electronic devices, books, notes are allowed during the exam.
- Show all work and justify all answers. Please sign the pledge below.

## Pledge of Honesty

I affirm that I will not give or receive any unauthorized help on this exam, and that all work will be my own.

Signature: \_\_\_\_\_

Part A		
QUESTION	VALUE	SCORE
1	14	
2	12	
3	10	
4	10	
5	12	
6	16	
7	16	
8	10	
TOTAL	100	

Part B		
QUESTION	VALUE	SCORE
9	14	
10	14	
11	12	
12	12	
13	12	
14	12	
15	12	
16	12	
TOTAL	100	

## Part A

1. (14 points) Consider the two planes  $4x + y - z = 4$  and  $x + 4y - z = 1$ .

(a) Find the (acute) angle between the planes.

$$\vec{n}_1 = \langle 4, 1, -1 \rangle$$

$$\vec{n}_2 = \langle 1, 4, -1 \rangle$$

$$\vec{n}_1 \cdot \vec{n}_2 = |\vec{n}_1| \cdot |\vec{n}_2| \cdot \cos \theta$$

$$4 + 4 + 1 = \sqrt{4^2 + 1^2 + (-1)^2} \cdot \sqrt{1^2 + 4^2 + (-1)^2} \cdot \cos \theta$$

$$\frac{9}{18} = \cos \theta$$

$$\frac{1}{2} = \cos \theta$$

$$\theta = 60^\circ$$

(b) Find a parametrization for the line of intersection of these two planes.

$$\vec{n}_1 \times \vec{n}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 4 & 1 & -1 \\ 1 & 4 & -1 \end{vmatrix} = 3\hat{i} + 3\hat{j} + 15\hat{k}$$

Take a point from the intersection:  $x = 0$

$$\begin{cases} y - z = 4 \\ 4y - z = 1 \end{cases}$$

$$-y + z = -4$$

$$4y - z = 1$$

---


$$3y = -3$$

$$y = -1$$

$$z = -5$$

$$\left. \begin{aligned} x(t) &= 0 + 3t \\ y(t) &= -1 + 3t \\ z(t) &= -5 + 15t \end{aligned} \right\}$$

A parametr. for the line of intersection.

2. (12 points) Find the distance between the two skew lines whose parametrizations are

$$l_1: \langle x, y, z \rangle = \langle 1, 0, 0 \rangle + t \langle -2, 1, 0 \rangle$$

$$l_2: \langle x, y, z \rangle = \langle 0, 1, 1 \rangle + s \langle 2, -1, 1 \rangle.$$

As  $l_1$  and  $l_2$  are skew, they are contained in two parallel planes. A normal to these 2 parallel planes is

$$\vec{n} = \langle -2, 1, 0 \rangle \times \langle 2, -1, 1 \rangle = \hat{i} + 2\hat{j} = \langle 1, 2, 0 \rangle.$$

Let  $s=0$  in  $l_2$ , we get the pt.  $(0, 1, 1)$  on  $l_2$ .

So, an eqn. of the plane containing  $l_2$  is

$$1 \cdot (x-0) + 2(y-1) + 0(z-1) = 0$$

$$x + 2y - 2 = 0$$

Let  $t=0$  in  $l_1$ , we get  $(1, 0, 0)$  on  $l_1$ .

$$\text{dist.} = \frac{|1 + 2 \cdot 0 - 2|}{\sqrt{1^2 + 2^2}} = \frac{1}{\sqrt{5}}$$

3. (10 points) A particle travels a total distance of  $26\pi$  along the parametric curve

$$\mathbf{r}(t) = (5 \sin t)\mathbf{i} + (5 \cos t)\mathbf{j} + 12t\mathbf{k}$$

from the starting point  $(0, -5, 12\pi)$ . Find its new location.

Notice when  $t = \pi$ , we get the pt.  $(0, -5, 12\pi)$ .

$$26\pi = \int_{\pi}^b |\vec{r}'(t)| dt.$$

$$\vec{r}'(t) = \langle 5 \cos t, -5 \sin t, 12 \rangle$$

$$|\vec{r}'(t)| = \sqrt{25 \cos^2 t + 25 \sin^2 t + 144} = 13$$

$$26\pi = \int_{\pi}^b 13 dt = 13t \Big|_{t=\pi}^{t=b} = 13(b - \pi)$$

$$2\pi = b - \pi$$

$$3\pi = b.$$

When  $t = 3\pi$ ,  $\vec{r}(3\pi) = 5 \sin(3\pi)\hat{i} + 5 \cos(3\pi)\hat{j} + 12(3\pi)\hat{k}$

$$\vec{r}(3\pi) = 0\hat{i} - 5\hat{j} + 36\pi\hat{k}$$

$$\text{New location} = (0, -5, 36\pi)$$

4. (10 points) Evaluate the limit

$$\lim_{(x,y) \rightarrow (0,0)} \frac{x^2 y^3}{x^4 + y^6}$$

or show that it does not exist.

5. (12 points) Find the linearization of the function

$$f(x, y) = \sqrt{12 - x^2 - 7y^2}$$

at  $(x, y) = (2, 1)$ , and then use it to find the approximate value of  $f(2.05, 0.98)$ .

$$f_x(x, y) = \frac{-2x}{2\sqrt{12 - x^2 - 7y^2}} = \frac{-x}{\sqrt{12 - x^2 - 7y^2}}$$

$$f_y(x, y) = \frac{-14y}{2\sqrt{12 - x^2 - 7y^2}} = \frac{-7y}{\sqrt{12 - x^2 - 7y^2}}$$

$$f_x(2, 1) = \frac{-2}{\sqrt{12 - 4 - 7}} = -2$$

$$f_y(2, 1) = \frac{-7}{\sqrt{12 - 4 - 7}} = -7$$

$$f(2, 1) = \sqrt{12 - 4 - 7} = 1$$

$$L(x, y) = 1 + (-2)(x - 2) + (-7)(y - 1)$$

$$L(2.05, 0.98) = 1 - 2 \cdot (0.05) - 7(-0.02)$$

$$L(2.05, 0.98) = 1.04 \approx f(2.05, 0.98)$$

6. (16 points) Find the critical points of the function

$$f(x, y) = x^2y - 8y^2 - x^2$$

and classify each of them as a local minimum, local maximum, or saddle point.

$$f_x = 2xy - 2x = 2x(y-1) \implies x=0 \text{ or } y=1.$$

$$f_y = x^2 - 16y$$

$x=0 : 0^2 - 16y = 0$   
 $y=0$   
 $(0,0)$

$y=1 :$   
 $x^2 - 16 \cdot 1 = 0$   
 $x = \pm 4$   
 $(-4, 1), (+4, 1)$

$$f_{xx} = 2y - 2$$

$$f_{xy} = 2x$$

$$f_{yy} = -16$$

$$D = -16(2y-2) - (2x)^2$$

$$\text{At } (0,0), \quad D(0,0) = -16(-2) - 0^2 = 32 > 0$$

$$f_{xx}(0,0) = -2 < 0 \quad \text{local max.}$$

$$\text{At } (-4,1), \quad D(-4,1) = -16(2-2) - (-8)^2 = -64 < 0.$$

saddle point.

$$\text{At } (4,1), \quad D(4,1) = -16(2-2) - 64 = -64 < 0$$

saddle pnt.

7. (16 points) Calculate the integral  $\iint_R y \, dA$  where  $R$  is the (finite) region lying between the curves  $x = y^2$  and  $y = x - 2$ .

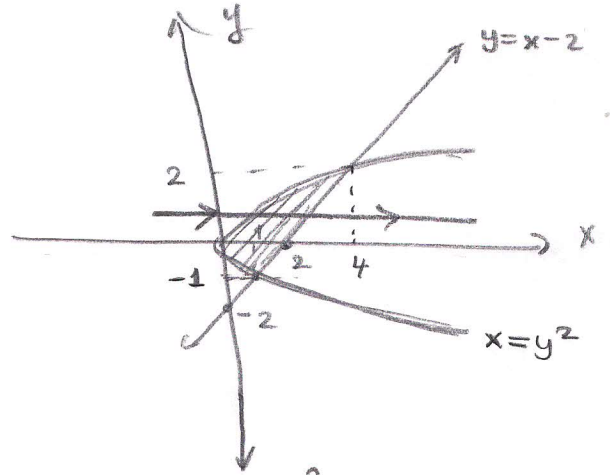
$$y + 2 = x$$

$$y + 2 = y^2$$

$$0 = y^2 - y - 2$$

$$0 = (y-2)(y+1)$$

$y=2$	$y=-1$
$x=4$	$x=1$



$$\iint_R y \, dA = \int_{-1}^2 \int_{y^2}^{y+2} y \, dx \, dy = \int_{-1}^2 \left[ yx \right]_{x=y^2}^{x=y+2} dy = \int_{-1}^2 (y(y+2) - y \cdot y^2) dy$$

$$= \int_{-1}^2 (y^2 + 2y - y^3) dy = \left. \frac{y^3}{3} + y^2 - \frac{y^4}{4} \right|_{y=-1}^{y=2} =$$

$$= \frac{8}{3} + 4 - \frac{16}{4} - \left[ \frac{-1}{3} + 1 - \frac{1}{4} \right] =$$

$$= \frac{9}{3} + 3 - 4 + \frac{1}{4} = 2 + \frac{1}{4} = \frac{9}{4}$$



8. (10 points) Circle the correct response for the following questions (no work is required, and there is no partial credit):

(a) The result obtained by reversing the order of integration in the iterated double integral

$$\int_0^1 \int_x^2 x \, dy \, dx$$

is

(i)  $\int_0^1 \int_x^2 x \, dx \, dy$

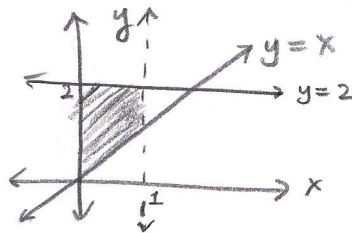
(iv)  $\int_0^1 \int_y^2 x \, dx \, dy + \int_1^2 \int_0^y x \, dx \, dy$

(ii)  $\int_0^2 \int_0^y x \, dx \, dy$

(v)  $\int_0^1 \int_0^y x \, dx \, dy + \int_1^2 \int_0^1 x \, dx \, dy$

(iii)  $\int_x^2 \int_0^1 x \, dx \, dy$

(vi)  $\int_0^1 \int_0^2 x \, dx \, dy + \int_1^2 \int_y^2 x \, dx \, dy$



(b) In polar coordinates, the integral

$$\int_{-2}^0 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} x \, dy \, dx$$

$\begin{matrix} \nearrow r \cos \theta \\ \searrow r \, dr \, d\theta \end{matrix}$

is

(i)  $\int_{\pi}^{2\pi} \int_0^2 r^2 \cos \theta \, dr \, d\theta$

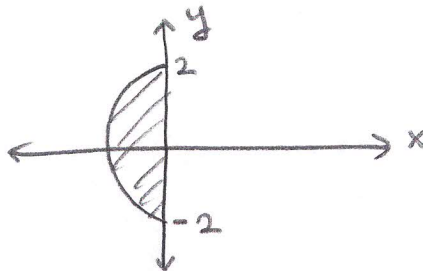
(iv)  $\int_{-\pi/2}^{\pi/2} \int_0^2 r \cos \theta \, dr \, d\theta$

(ii)  $\int_{\pi/2}^{3\pi/2} \int_0^2 r^2 \cos \theta \, dr \, d\theta$

(v)  $\int_{\pi/2}^{3\pi/2} \int_0^2 r \cos \theta \, dr \, d\theta$

(iii)  $\int_0^{2\pi} \int_0^2 r^2 \cos \theta \, dr \, d\theta$

(vi)  $\int_0^{2\pi} \int_0^2 r \cos \theta \, dr \, d\theta$



## Part B

9. (14 points) Evaluate the triple integral  $\iiint_E x \, dV$ , where  $E$  is the solid tetrahedron bounded by the four planes  $x = 0$ ,  $y = 0$ ,  $z = 0$  and  $x + y + z = 1$ , whose vertices are  $(0, 0, 0)$ ,  $(1, 0, 0)$ ,  $(0, 1, 0)$ , and  $(0, 0, 1)$ .

$$\iiint_E x \, dV = \int_0^1 \int_0^{1-x} \int_0^{1-x-y} x \, dz \, dy \, dx = \int_0^1 \int_0^{1-x} x(1-x-y) \, dy \, dx =$$

$$= \int_0^1 \int_0^{1-x} (x - x^2 - xy) \, dy \, dx = \int_0^1 \left( xy - x^2 y - \frac{xy^2}{2} \Big|_{y=0}^{y=1-x} \right) dx =$$


$$= \int_0^1 \left[ x(1-x) - x^2(1-x) - x \cdot \frac{(1-x)^2}{2} \right] dx = \int_0^1 \left[ x - x^2 - x^2 + x^3 - \frac{x(1-x)^2}{2} \right] dx =$$

$$= \int_0^1 \left[ x - 2x^2 + x^3 - \frac{x}{2} + \frac{x^3}{2} + x^2 \right] dx = \int_0^1 \left( \frac{x}{2} - x^2 + \frac{x^3}{2} \right) dx =$$

$$= \frac{x^2}{4} - \frac{x^3}{3} + \frac{x^4}{8} \Big|_{x=0}^{x=1} = \frac{1}{4} - \frac{1}{3} + \frac{1}{8} = \frac{1}{24}$$

10. (14 points) Evaluate the integral

$$x^2 + y^2 + z^2 = \rho^2$$

$$I = \int_0^1 \int_0^{\sqrt{1-x^2}} \int_{-\sqrt{1-x^2-y^2}}^{\sqrt{1-x^2-y^2}} \underbrace{\sqrt{x^2+y^2+z^2}}_{\rho^2 \sin \phi} dz dy dx.$$


$$I = \int_0^{\pi} \int_0^{\pi/2} \int_0^1 \sqrt{\rho^2} \rho^2 \sin \phi \, d\rho \, d\theta \, d\phi = \int_0^{\pi} \int_0^{\pi/2} \int_0^1 \rho^3 \sin \phi \, d\rho \, d\theta \, d\phi =$$

$$= \int_0^{\pi} \int_0^{\pi/2} \left( \frac{\rho^4}{4} \sin \phi \Big|_{\rho=0}^{\rho=1} \right) d\theta \, d\phi = \int_0^{\pi} \int_0^{\pi/2} \frac{1}{4} \sin \phi \, d\theta \, d\phi$$

$$= \int_0^{\pi} \frac{\pi}{2} \cdot \frac{1}{4} \cdot \sin \phi \, d\phi = \frac{\pi}{8} \cdot (-\cos \phi) \Big|_{\phi=0}^{\phi=\pi} = -\frac{\pi}{8} (-1 - 1) =$$

$$= \frac{2\pi}{8} = \frac{\pi}{4}$$

11. (12 points) Evaluate the line integral  $\int_C (x - y + z + 2) ds$ , where  $C$  is the straight line segment from  $(0, 1, 1)$  to  $(1, 0, 1)$ .

$$\vec{V} = (1, 0, 1) - (0, 1, 1) = \langle 1, -1, 0 \rangle$$

A param. for  $C$  :

$$\begin{aligned} x(t) &= 0 + t \\ y(t) &= 1 - t \\ z(t) &= 1 \end{aligned}$$

when  $t=0$ , we have  $(0, 1, 1)$

when  $t=1$ , "  $(1, 0, 1)$

$$\vec{r}(t) = \langle t, 1-t, 1 \rangle$$

$$\vec{r}'(t) = \langle 1, -1, 0 \rangle$$

$$\int_C (x - y + z + 2) ds = \int_0^1 [t - (1-t) + 1 + 2] \cdot \sqrt{1^2 + (-1)^2 + 0^2} dt =$$

$$\int_0^1 [t - 1 + t + 3] \sqrt{2} dt = \sqrt{2} \int_0^1 (2t + 2) dt = \sqrt{2} (t^2 + 2t) \Big|_{t=0}^{t=1}$$

$$= \sqrt{2} (1 + 2) = 3\sqrt{2}.$$

12. (12 points) Let  $f$  be a scalar function and  $\mathbf{F}$  be a vector field. For each expression, identify whether it is a scalar function, a vector field, or nonsense by circling the appropriate response (no work is required, and there is no partial credit).

Note that  $\text{grad}(f) = \nabla f$  denotes the gradient of  $f$ .

---

• $\text{curl}(f)$	Scalar function	Vector field	<u>Nonsense</u>
• $\text{grad}(f)$	Scalar function	<u>Vector field</u>	Nonsense
• $\text{div}(\mathbf{F})$	<u>Scalar function</u>	Vector field	Nonsense
• $\text{curl}(\text{grad}(f))$	Scalar function	<u>Vector field</u>	Nonsense
• $\text{grad}(\mathbf{F})$	Scalar function	Vector field	<u>Nonsense</u>
• $\text{grad}(\text{div}(\mathbf{F}))$	Scalar function	<u>Vector field</u>	Nonsense
• $\text{div}(\text{grad}(f))$	<u>Scalar function</u>	Vector field	Nonsense
• $\text{grad}(\text{div}(f))$	Scalar function	Vector field	<u>Nonsense</u>
• $\text{curl}(\text{curl}(\mathbf{F}))$	Scalar function	<u>Vector field</u>	Nonsense
• $\text{div}(\text{div}(\mathbf{F}))$	Scalar function	Vector field	<u>Nonsense</u>
• $(\text{grad}(f)) \times (\text{div}(\mathbf{F}))$	Scalar function	Vector field	<u>Nonsense</u>
• $\text{div}(\text{curl}(\text{grad}(f)))$	<u>Scalar function</u>	Vector field	Nonsense

13. (12 points) Find all possible values of the constants  $a$  and  $b$  such that the vector field

$$\mathbf{F}(x, y, z) = (2bxz^3 + ayz + 2xy)\mathbf{i} + (2by + ax^2 + axz)\mathbf{j} + (axy + bz + bx^2z^2)\mathbf{k}$$

is conservative (i.e., that the work done by the field on a particle moving through space does not depend on the particle's path).

Since  $\vec{F}$  is conservative, there exists  $f$  such that  $\vec{\nabla}f = \vec{F}$ :

$$f_x(x, y, z) = 2bxz^3 + ayz + 2xy$$

$$f_y(x, y, z) = 2by + ax^2 + axz$$

$$f_z(x, y, z) = axy + bz + bx^2z^2$$

$$\left. \begin{aligned} f_{xy}(x, y, z) &= az + 2x \\ f_{yx}(x, y, z) &= 2ax + az \end{aligned} \right\} \Rightarrow \boxed{\begin{aligned} 2a &= 2 \\ a &= 1 \end{aligned}}$$

$$\left. \begin{aligned} f_{xz}(x, y, z) &= 6bxz^2 + ay \\ f_{zx}(x, y, z) &= ay + 2bxz^2 \end{aligned} \right\} \Rightarrow \begin{aligned} 6bxz^2 &= 2bxz^2 \\ 2b &= 6b \\ \boxed{b} &= 0 \end{aligned}$$

$$\left. \begin{aligned} f_{yz}(x, y, z) &= ax \\ f_{zy}(x, y, z) &= ax \end{aligned} \right\} \checkmark$$

14. (12 points) Compute the work integral  $\oint_C \mathbf{F} \cdot d\mathbf{r}$ , where

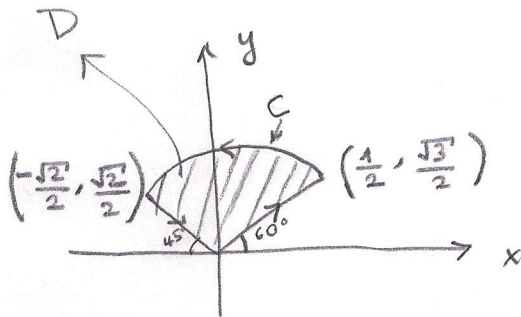
$$\mathbf{F}(x, y) = \underbrace{(x^{2015} + x^2y)}_P \mathbf{i} + \underbrace{(xy^2 + 2e^y)}_Q \mathbf{j}$$

and the closed path  $C$ , oriented counterclockwise, consists of the following three pieces:

$C_1$  : the line segment from  $(0, 0)$  to  $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ ,

$C_2$  : the curve  $y = \sqrt{1-x^2}$  with  $-\frac{\sqrt{2}}{2} \leq x \leq \frac{1}{2}$ ,

$C_3$  : the line segment from  $\left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$  to  $(0, 0)$ .



Using Green's theorem:

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_D \left[ \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right] dA =$$

$$= \iint_D [y^2 - x^2] dA = \int_{\frac{\pi}{3}}^{\frac{3\pi}{4}} \int_0^1 [r^2 \sin^2 \theta - r^2 \cos^2 \theta] r dr d\theta =$$

$$y = r \sin \theta$$

$$x = r \cos \theta$$

$$= \int_{\frac{\pi}{3}}^{\frac{3\pi}{4}} \int_0^1 r^3 (-\cos 2\theta) dr d\theta = - \int_0^1 r^3 dr \cdot \int_{\frac{\pi}{3}}^{\frac{3\pi}{4}} \cos 2\theta d\theta =$$

$$= \left( -\frac{r^4}{4} \Big|_0^1 \right) \cdot \left( \frac{\sin 2\theta}{2} \Big|_{\theta=\frac{\pi}{3}}^{\theta=\frac{3\pi}{4}} \right) = -\frac{1}{4} \cdot \frac{1}{2} \cdot \left[ \sin \frac{3\pi}{2} - \sin \frac{2\pi}{3} \right] =$$

$$= -\frac{1}{8} \left[ -1 - \frac{\sqrt{3}}{2} \right] = \frac{1}{8} + \frac{\sqrt{3}}{16}$$

15. (12 points) Find the surface area of the part of the paraboloid  $z = x^2 + y^2$  that lies under the plane  $z = 4$ .

$$\begin{aligned} A &= \iint_D \sqrt{1 + (2x)^2 + (2y)^2} \, dA = \iint_D \sqrt{1 + 4(x^2 + y^2)} \, dA = \\ &= \int_0^{2\pi} \int_0^2 \sqrt{1 + 4r^2} \, r \, dr \, d\theta = \int_0^{2\pi} d\theta \cdot \int_0^2 r \cdot \sqrt{1 + 4r^2} \, dr = \\ &= 2\pi \cdot \frac{1}{8} \cdot \frac{2}{3} \cdot (1 + 4r^2)^{3/2} \Big|_{r=0}^{r=2} = \frac{\pi}{6} \cdot [(17)^{3/2} - 1]. \end{aligned}$$



16. (12 points) Compute the flux of the vector field

$$\mathbf{F}(x, y, z) = (-3x^2y)\mathbf{i} + (z - y)\mathbf{j} + (2x)\mathbf{k}$$

through the surface  $S$  given by the part of the plane  $z = 1 + 2x + y$  lying above the rectangle  $0 \leq x \leq 2$ ,  $0 \leq y \leq 3$ , with upward orientation.

$$\vec{F}(x, y, z) = (-3x^2y)\hat{i} + (z - y)\hat{j} + (2x)\hat{k}$$

$$x = u$$

$$y = v$$

$$z = 1 + 2u + v$$

$$\vec{r}(u, v) = \langle u, v, 1 + 2u + v \rangle$$

$$\vec{r}_u = \langle 1, 0, 2 \rangle$$

$$\vec{r}_v = \langle 0, 1, 1 \rangle$$

$$\vec{r}_u \times \vec{r}_v = \hat{k} - 2\hat{i} - \hat{j} = \langle -2, -1, 1 \rangle$$

$$\text{Flux} = \int_S \vec{F} \cdot d\mathbf{S} = \iint_D \vec{F} \cdot (\vec{r}_u \times \vec{r}_v) dA =$$

$$= \int_0^3 \int_0^2 (-3u^2v\hat{i} + (1+2u)\hat{j} + 2u\hat{k}) \cdot (-2\hat{i} - \hat{j} + \hat{k}) du dv$$

$$= \int_0^3 \int_0^2 (6u^2v - 1 - 2u + 2u) du dv = \int_0^3 \left[ \frac{6u^3v}{3} - u \right]_{u=0}^{u=2} dv =$$

$$= \int_0^3 (2 \cdot 2^3 \cdot v - 2) dv = \int_0^3 (16v - 2) dv = (8v^2 - 2v) \Big|_{v=0}^{v=3} =$$

$$= 72 - 6 = 66$$