Math 164: Multidimensional Calculus

Midterm 1

October 11, 2016

NAME (please print legibly): ______________________________

Your University ID Number: ______________________________

Indicate your instructor with a check in the appropriate box:

Kleene	TR 12:30-1:45pm	
Salur	MW 3:25-4:40pm	
Gafni	TR 3:25-4:40pm	
Lee	MWF 09:00-09:50am	

- You have 75 minutes to work on this exam.
- You are responsible for checking that this exam has all 9 pages.
- No calculators, phones, electronic devices, books, notes are allowed during the exam.
- Show all work and justify all answers.
- Please sign the pledge below.

Pledge of Honesty

I affirm that I will not give or receive any unauthorized help on this exam, and that all work will be my own.

Signature: ______________________________

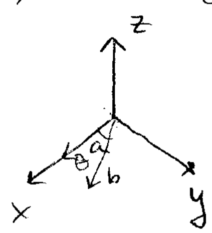
QUESTION	VALUE	SCORE
1	15	
2	20	
3	15	
4	10	
5	15	
6	15	
7	10	
TOTAL	100	

1. (15 points) Consider the vectors $a = < 1, 0, 0 >$, $b = < 1, 0, -1 >$ and $c = < 1, 2, -1 >$.

(a) Compute the scalar projection of b onto c.

$$\mathrm{Comp}_c(b) = \frac{b \cdot c}{|c|} = \frac{1+0+1}{\sqrt{1+4+1}} = \frac{2}{\sqrt{6}}$$

(b) Find the angle between a and b.



$$a \cdot b = |a||b|\cos\theta \Rightarrow 1 = 1 \cdot \sqrt{2} \cdot \cos\theta$$

$$\cos\theta = \frac{1}{\sqrt{2}} \Rightarrow \theta = \frac{\pi}{4}$$

(c) A vector $v = < 5, y, z >$ satisfies $a \cdot v = b \cdot v = c \cdot v$. Find y and z.

$$\left.\begin{array}{l} \left.\begin{array}{l} a \cdot V = 5 \\ b \cdot V = 5 - z \end{array}\right\} \Rightarrow z = 0 \\ c \cdot V = 5 + 2y - z \end{array}\right\} \Rightarrow y = 0$$

$$\vec{V} = \langle 5, 0, 0 \rangle$$

2. (20 points) Consider the four points $P(0,1,5), Q(1,2,8), R(2,-1,0)$, and $S(1,2,3)$

(a) Find an equation for the plane that passes through points P, Q, and R.

$$\left.\begin{aligned}\vec{a} &= \vec{PQ} = \langle 1,1,3\rangle \\ \vec{b} &= \vec{PR} = \langle 2,-2,-5\rangle\end{aligned}\right\} \Rightarrow \vec{a}\times\vec{b} = \begin{vmatrix} i & j & k \\ 1 & 1 & 3 \\ 2 & -2 & -5 \end{vmatrix} = i(-5+6) - j(-5-6) + k(-2-2)$$

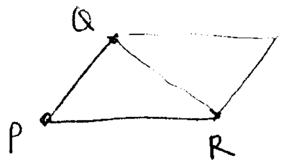
$$\boxed{\vec{n} = i + 11j - 4k}$$

$$\vec{n}\cdot\langle x-0, y-1, z-5\rangle = 0$$

$$1(x-0) + 11(y-1) - 4(z-5) = 0$$

$$\boxed{x + 11(y-1) - 4(z-5) = 0}$$

(b) Find the area of the triangle with vertices P, Q, and R.



area of $\triangle = \frac{1}{2}$ area of $▱$ $= \frac{1}{2}|a|\cdot|b|\sin\theta$

$= \frac{1}{2}|a\times b|$

$= \frac{1}{2}\sqrt{1+121+16}$

$= \boxed{\frac{1}{2}\sqrt{138}}$

(c) Find the volume of the parallelepiped determined by P, Q, R, and S.

$$\text{vol}(\;) = |c\cdot(a\times b)| = |\langle 1,1,-2\rangle\cdot\langle 1,11,-4\rangle| = |1+11+8|$$

$$\vec{c} = \vec{PS} = \langle 1,1,-2\rangle \qquad = \boxed{20}$$

(d) Find the distance from S to the plane determined by P, Q, and R.

$$D = \frac{20}{\sqrt{138}} = \boxed{\frac{20}{\sqrt{138}}} \quad \text{since volume} = D\cdot\text{area of } ▱$$

3. (15 points) Let ℓ_1 be the line that passes through the points $(1, -2, 3)$ and $(2, 0, -1)$, and let ℓ_2 be the line that passes through the point $(3, 1, 2)$ and is perpendicular to the plane $x + 2y + 4z = 0$.

(a) Find symmetric and parametric equations for ℓ_1. Clearly label each set as symmetric or parametric.

$$\ell_1 : \boxed{\left\{\begin{array}{l} x = 1 + t \\ y = -2 + 2t \\ z = 3 - 4t \end{array}\right\}} \text{ parametric, } t \in \mathbb{R} \qquad \vec{v}_1 = \langle 1, 2, -4 \rangle$$

$$\boxed{x - 1 = \frac{y+2}{2} = \frac{z-3}{-4}} \text{ symmetric}$$

(b) Find symmetric and parametric equations for ℓ_2. Clearly label each set as symmetric or parametric. parametric

$$\ell_2 : \left\{\begin{array}{l} x = 3 + s \\ y = 1 + 2s \\ z = 2 + 4s \end{array}\right\} s \in \mathbb{R} \qquad \vec{v}_2 = \vec{n} = \langle 1, 2, 4 \rangle$$

$$x - 3 = \frac{y-1}{2} = \frac{z-2}{4}$$

(c) Are these lines intersecting, parallel, or skew? Briefly explain your answer.

$$\left.\begin{array}{l} 1 + t = 3 + s \Rightarrow t = s + 2 \Rightarrow t - s = 2 \\ -2 + 2t = 1 + 2s \Rightarrow 2(t - s) = 3 \end{array}\right\} \Rightarrow 4 = 3 \text{ ※ no sol'n}$$

• not parallel since $\vec{v}_1$ not parallel to $\vec{v}_2$

• not intersecting because even the first two simultaneous equations in t & s give no solution.

$\boxed{\text{skew}}$

4. (10 points) Consider the curve $\mathbf{r}(t) = 2t\mathbf{i} + (3-t)\mathbf{j} - 2t\mathbf{k}$.

(a) Find the arc length of the curve between $t = 0$ and $t = 1$.

$$AL = \int_0^1 \sqrt{(2)^2 + (-1)^2 + (-2)^2}\, dt = \int_0^1 \sqrt{9}\, dt = 3t\Big]_0^1 = 3$$

$$r'(t) = 2i - j - 2k$$

$$|r'(t)| = \sqrt{(2)^2 + (-1)^2 + (-2)^2}$$

(b) Reparametrize the curve in terms of arc length s measured from $t = 0$ in the direction of increasing t.

$$s(t) = \int_0^t |r'(t)|\, dt = \int_0^t 3\, dt = 3t \Rightarrow t = t(s) = \frac{s}{3}$$

$$\boxed{r'(t) = 2i + (-1)j + (-1)k}$$

so $\vec{r}(t) = 2\left(\frac{s}{3}\right)\vec{i} + \left(3 - \frac{s}{3}\right)\vec{j} - 2\left(\frac{s}{3}\right)\vec{k}$

$$\boxed{\vec{r}(t) = \frac{2}{3}s\vec{i} + \left(3 - \frac{s}{3}\right)\vec{j} - \frac{2}{3}s\vec{k}}$$

5. (15 points) Consider the curve $\mathbf{r}(t) = \cos 2t\mathbf{i} + \sin 2t\mathbf{j} + 2t\mathbf{k}$.

(a) Find the unit tangent vector $\mathbf{T}(t)$.

$$\vec{T}(t) = \frac{r'(t)}{|r'(t)|} = \frac{-2\sin(2t)\vec{i} + 2\cos 2t\,\vec{j} + 2\vec{k}}{\sqrt{4\sin^2(2t) + 4\cos^2 2t + 4}}$$

$$= \frac{-2\sin(2t)\vec{i} + 2\cos 2t\,\vec{j} + 2\vec{k}}{\sqrt{4+4}}$$

$$= \frac{1}{\sqrt{2}}\left(-\sin(2t)\vec{i} + \cos(2t)\vec{j} + \vec{k}\right)$$

(b) Find the curvature $\kappa(t)$. Recall the curvature formula $\kappa(t) = \frac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|}$.

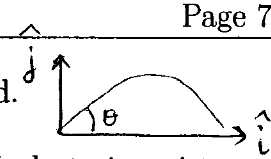
$$\kappa(t) = \frac{\sqrt{2}}{2\sqrt{2}} = \boxed{\frac{1}{2}}$$

$$T'(t) = \frac{1}{\sqrt{2}}\left(-2\cos(2t)\vec{i} - 2\sin(2t)\vec{j} + 0\vec{k}\right)$$

$$|T'(t)| = \frac{1}{\sqrt{2}}\sqrt{4(\cos^2(4t) + \sin^2(4t))} = \frac{2}{\sqrt{2}} = \boxed{\sqrt{2}}$$

$$|r'(t)| = \boxed{2\sqrt{2}}$$

6. (15 points) A gun has a muzzle speed of 80 meters per second.



(a) Assuming projectile motion, find the position function $r(t)$. Neglect air resistance and use $g = 9.8m/sec^2$ as the acceleration of gravity.

$$\vec{a}(t) = 0\hat{i} - 9.8\hat{j} = \langle 0, -9.8 \rangle$$

$$\vec{v}(t) = \langle 0, -9.8 \rangle t + \langle 80\cos\theta, 80\sin\theta \rangle$$

init. pos.

$$\boxed{\vec{r}(t) = \langle 0, -4.9 \rangle t^2 + \langle 80\cos\theta, 80\sin\theta \rangle t} + \langle 0, 0 \rangle$$

(b) What angle of elevation should be used to hit an object 200 meters away?

$r_x = 80\cos\theta\, t$ and $r_y = -4.9t^2 + 80\sin\theta\, t = 0$

when $t = 0$ and $t = \frac{80\sin\theta}{4.9}$

so $200 = \frac{80\cos\theta \cdot 80\sin\theta}{4.9}$

$$\frac{200 \cdot 4.9}{80^2} = \sin\theta\cos\theta = \frac{1}{2}\sin 2\theta$$

$$\frac{400 \cdot 4.9}{80^2} = \sin 2\theta$$

$$\boxed{\frac{1}{2}\arcsin\left(\frac{400 \cdot 4.9}{80^2}\right) = \theta}$$

7. (10 points) Evaluate the limit or show that it does not exist.

(a) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{4xy}{x^2+2y^2}$

through $x=0$, get 0

through $x=y$, get $\frac{4x^2}{3x^2} \Rightarrow \frac{4}{3} \neq 0$

$\boxed{\text{DNE}}$

(b) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^3+5y^3}{x^2+y^2}$

Hint: Use polar coordinates: $x = r\cos\theta$, $y = r\sin\theta$.

$$\frac{r^3\cos^3\theta + 5r^3\sin^3\theta}{r^2(\cancel{\cos^2\theta + \sin^2\theta})} = \frac{r^3\cos^3\theta + 5r^3\sin^3\theta}{r^2} =$$

$$r\cos^3\theta + 5r\sin^3\theta = r\underbrace{(\cos^3\theta + 5\sin^3\theta)}_{\substack{\text{bounded}\\ \text{for all } \theta}} \longrightarrow \boxed{0} \text{ as } r\to 0^+$$

Blank page for scratch work