UNIVERSITY of ROCHESTER

Math 164: Multidimensional Calculus

Final Exam Solutions

May 5, 2008

Name (please print legibly): ____

University ID Number: ____

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- Calculators, cell phones, iPods, and other electronics are not allowed on this exam.
- Please show all your work. You may use the backs of pages if necessary. A correct answer with no work shown will not receive full credit. Please label and circle your final answers.
- You are responsible for checking that this exam has all 17 pages. Please tell us immediately if your exam is missing a page. Missing pages will not contribute to your total score.

Question	Points	Score
1	20	
2	10	
3	24	
4	12	
5	12	
6	12	
7	12	
8	10	
9	12	
10	10	
11	12	
12	10	
13	12	
14	20	
15	12	
Total:	200	

1. (20 points)

(a) **(10 points)** Write an equation of the plane tangent to the ellipsoid $2x^2 + 4y^2 + z^2 = 45$ at the point $(2, -3, -1)$.

Solution. Write $F(x, y, z) = 2x^2 + 4y^2 + z^2 - 45$. THen $F(x, y, z) = 0$ is an equation of the ellipsoid. By Equation 18 in Section 14.6, page 948, a vector normal to the ellipsoid surface at (x, y, z) is $\nabla F(x, y, z) = 4x\mathbf{i} + 8y\mathbf{j} + 2z\mathbf{k}$, so $\nabla F(x, -3, -1) = 8\mathbf{i} - 24\mathbf{j} - 2\mathbf{k}$ is normal to the ellipsoid at $(2, -3, -1)$. Equation 19 in Section 14.6, page 948, then gives the answer in the form $8(x - 2) - 24(y - 3) - 2(z + 1) = 0$ or $4x - 12y - z = 45$.

(b) **(10 points)** Find all the points on the ellipsoid $2x^2 + 4y^2 + z^2 = 45$ at which the tangent plane is parallel to the plane $4y + z = 8$.

Solution. We find all points (x, y, z) that satisfy $\langle 4x, 8y, 2z\rangle = a\langle 0, 4, 1\rangle$, for some constant a. Since (x, y, z), with $x = 0$, $y = a/2$, and $z = a/2$, lies on the ellipsiod, we substitute these into $2x^2 + 4y^2 + z^2 = 45$ to find that $a = \pm 6$. Hence the required points are $(0, 3, 3)$ and $(0, -3, -3)$.

2. (10 points) Locate and classify the critical points of the function $f(x, y) = 3x - x^3 - 3xy^2$.

Solution. f is a polynomial, so all its partial derivatives exist and are continuous everywhere. Now $f_x(x, y) = 3 - 3x^2 - 3y^2 = 0$ and $f_y(x, y) = -6xy = 0$ give us the four critical points $(1, 0)$, $(-1, 0)$, $(0, 1)$, and $(0, -1)$. We find that $f_{xx}(x, y) = -6x$, $f_{xy}(x, y) = -6y$, and $f_{yy}(x, y) = -6x$. Applying Theorem 3 in Section 14.7, page 954, we find that at $(1, 0)$ there is a local maximum, at $(-1, 0)$ there is a local minimum, and at $(0, 1)$ and $(0, -1)$ there are saddle points.

3. (24 points)

(a) **(14 points)** Show that $\mathbf{F}(x, y, z) = (2xy^3 + z^2)\mathbf{i} + (3x^2y^2 + 2yz)\mathbf{j} + (y^2 + 2xz)\mathbf{k}$ is a conservative vector field. Then find a potential function f such that $\mathbf{F} = \nabla f$.

Solution. curl $\mathbf{F} = (2y - 2y)\mathbf{i} + (2z - 2z)\mathbf{j} + (6xy^2 - 6xy^2)\mathbf{k} = bf0$ and the domain of $\mathbf{F}$ is $\mathbb{R}^3$, so $\mathbf{F}$ is conservative, by Theorem 4 in Section 16.5, page 1092. Thus there is a potential function f such that $\triangledown f = \mathbf{F}$. Then $f_x(x, y, z) = 2xy^3 + z^2$ implies $f(x, y, z) = x^2y^3 + xz^2 + g(y, z)$ and $f_y(x, y, z) = 3x^2y^2 + g_y(y, z)$. But $f_y(x, y, z) = 3x^2y^2 + 2yz$ so $g(y, z) = y^2z + h(z)$. Then $f(x, y, z) = x^2y^3 + xz^2 + h(z)$ implies $f_z(x, y, z) = 2xz + y^2 + h'(z)$. But $f_z(x, y, z) = y^2 + 2xz$. So $h'(z) = 0$. Hence a potential function f for $\mathbf{F}$ is $f(x, y, z) = x^2y^3 + xz^2 + y^2z + K$, where K is some constant.

(b) **(10 points)** Let C be the arc of the curve $y = x \sin x$ from $(0, 0)$ to $(\pi, 0)$. Use the function f from part (a) to evaluate the integral

$$\int_C \mathbf{F} \cdot d\mathbf{r}.$$

Solution. Since $\mathbf{F}$ is conservative and since curl $\mathbf{F} = 0$, by part (a), we have $\int_C \mathbf{F} \cdot d\mathbf{r} = 0$. We may also see this by evaluating the function f at the endoints – we obtain $0 - 0 = 0$ in this way (see Theorem 2 in Section 16.3, page 1074).

4. (12 points) Use Green's Theorem to evaluate the line integral

$$\int_C (xy\,dx + x^2\,dy),$$

where C is the triangle with vertices $(0,0)$, $(1,0)$, and $(1,2)$.

Solution. Sketch a graph of the triangle C. Note that C is traversed counterclockwise and that C is comprised of the three line segments C_1, C_2, and C_3 defined as follows. $C_1 : 0 \le x \le 1$, $y = 0$; $C_2 : x = 1$, $0 \le y \le 2$; $C_3 : x = x$, $y = 2x$, and x ranges from $x = 1$ to $x = 0$. Using Green's Theorem, we find that

$$\begin{aligned}\int_C (xy\,dx + x^2\,dy) &= \iint_D \left[\frac{\partial}{\partial x}(x^2) - \frac{\partial}{\partial y}(xy)\right] dA \\ &= \iint_D (2x - x)\,dA \\ &= \int_0^1 \int_0^{2\pi} x\,dy\,dx \\ &= \frac{2}{3}\end{aligned}$$

To verify Green's Theorem, we evaluate the line integral directly:

$$\int_C (xy\,dx + x^2\,dy) = \int_0^1 0\,dx + \int_0^2 (0+1)\,dy + \int_1^0 (2x^2 + 2x^2)\,dx = \frac{2}{3}.$$

5. (12 points) Let D be the lamina bounded by $x = 0$ and $x = 4 - y^2$ and constant density function $\rho(x, y) = 1$. Find the center of mass of D.

Solution. By symmetry of the lamina $D\{(x, y)|0 \leq x \leq 4 - y^2, -2 \leq y \leq 2\}$, its center of mass is given by $(\bar{x}, \bar{y}) = (\bar{x}, 0)$. We find

$$\bar{x} = \frac{\iint_D x\rho(x, y)\, dA}{\iint_D \rho(x, y)\, dA} = \frac{\iint_D x\, dA}{\iint_D dA},$$

where, by a short calculation,

$$\iint_D dA = \int_{-2}^{2} \int_{0}^{4-y^2} x\, dx\, dy = \frac{256}{15}$$

and that

$$\iint_D dA = \int_{-2}^{2} \int_{0}^{4-y^2} dx\, dy = \frac{32}{3},$$

so that $\bar{x} = \dfrac{8}{5}$.

6. (12 points) Find the area of the part of the surface $z = x^2 + 2y$ that lies above the triangle with vertices $(0, 0)$, $(1, 0)$, and $(1, 2)$.

Solution. $z = f(x, y) = x^2 + 2y$ with $0 \leq x \leq 1$ and $0 \leq y \leq 2x$. Thus

$$\begin{aligned} A(S) &= \iint_D \sqrt{1 + 4x^2 + 4}\, dA \\ &= \int_0^1 \int_0^{2x} \sqrt{5 + 4x^2}\, dy\, dx \\ &= \int_0^1 2x\sqrt{5 + 4x^2}\, dx \\ &= \left[\frac{1}{6}(5 + 4x^2)^{3/2} \right]_0^1 \\ &= \frac{1}{6}(27 - 5\sqrt{5}) \end{aligned}$$

7. (12 points) Evaluate the surface integral

$$\iint_S (x^2z + y^2z)\, dS$$

where S is the part of the plane $z = 4 + x + y$ that lies inside the cylinder $x^2 + y^2 = 4$.

Solution. $z = f(x, y) = 4 + x + y$ with $0 \le x^2 + y^2 \le 4$. So $r_{\mathbf{x}} \times r_{\mathbf{y}} = -\mathbf{i} - \mathbf{j} + \mathbf{k}$. Then

$$\begin{aligned}
\iint_S (x^2z + y^2z)\, dS &= \iint_{x^2+y^2\le 4} (x^2 + y^2)(4 + x + y)\sqrt{3}\, dA \\
&= \int_0^2 \int_0^{2\pi} \sqrt{3}r^3(4 + r\cos\theta + r\sin\theta)\, d\theta\, dr \\
&= \int_0^2 8\pi\sqrt{3}r^3\, dr \\
&= 32\pi\sqrt{3}
\end{aligned}$$

8. (10 points) Evaluate the flux integral

$$\iint_S \mathbf{F} \cdot \mathbf{n}\, dS$$

where $\mathbf{F}(x, y, z) = x^2\mathbf{i} + xy\mathbf{j} + z\mathbf{k}$ and S is the part of the paraboloid $z = x^2 + y^2$ below the plane $z = 1$ with upward orientation.

Solution. We use Formula 9 on page 1117. Note that $z = f(x, y) = x^2 + y^2$, $r_{\mathbf{x}} \times r_{\mathbf{y}} = -2x\mathbf{i} - 2y\mathbf{j} + \mathbf{k}$ (because of upward orientation), and $\mathbf{F}(\mathbf{r}(x, y)) \cdot (r_{\mathbf{x}} \times r_{\mathbf{y}}) = -2x^3 - 2xy^2 + x^2 + y^2$. Then

$$\begin{aligned}
\iint_S \mathbf{F} \cdot \mathbf{n}\, dS &= \iint_D \mathbf{F} \cdot (r_{\mathbf{x}} \times r_{\mathbf{y}})\, dA \\
&= \iint_{x^2+y^2 \le 1} (-2x^3 - 2xy^2 + x^2 + y^2)\, dA \\
&= \int_0^1 (-2r^3 \cos^3\theta - 2r^2 \cos\theta \sin^2\theta + r^2) r\, dr\, d\theta \\
&= \int_0^1 r^3 (2\pi)\, dr = \frac{\pi}{2}
\end{aligned}$$

9. (12 points) Use the Divergence Theorem to calculate the flux integral

$$\iint_S \mathbf{F} \cdot \mathbf{n}\, dS$$

where S is the surface of the sphere of radius and $\mathbf{F} = x\mathbf{i} + e^z\mathbf{j} + 3z\mathbf{k}$.

Solution. Using div $\mathbf{F} = 5$, we find that

$$\iint_S \mathbf{F} \cdot \mathbf{n}\, dS = \iint_E \operatorname{div} \mathbf{F}\, dV = 5 \iint_E dV = 5\left(\frac{4}{3}\pi \cdot 2^3\right) = \frac{160\pi}{3},$$

where E is the solid enclosed by the surface S.

10. (10 points) Find the volume of the region below the plane $x + 2y + 3z = 6$ in the first octant.

Solution. Let V denote the volume of the region below the plane $x + 2y + 3z = 6$ in the first octant. Let $z = f(x, y) = 6 - 2x - 2y$. Now, if $z = 0$ then $D = \{(x, y)|0 \leq x \leq 3, 0 \leq y \leq 3 - x\}$. We compute that

$$V = \iint_D dA = \int_0^3 \int_0^{3-x} (6 - 2x - 2y)\, dy\, dx = \int_0^3 (x^2 - 6x + 9)\, dx = 9.$$

11. (12 points) Let D be the region defined by the inequalities $0 \le x \le 1$ and $\sqrt[3]{x} \le y \le 1$. Evaluate the integral

$$\iint_D e^{y^4}\, dA$$

Solution. We compute that

$$\iint_D e^{y^4}\, dA = \int_0^1 \int_0^{y^3} e^{y^4}\, dx\, dy = \int_0^1 y^3 e^{y^4}\, dy = \frac{e-1}{4}.$$

12. (10 points) Find the length of the arc parametrized by

$$\mathbf{r}(t) = \cos 2t\mathbf{i} + \sin 2t\mathbf{j} + 3t\mathbf{k}$$

where $\pi \leq t \leq 4\pi$.

Solution. Let s denote the arclength of the arc parametrized by $\mathbf{r}(t) = \cos 2t\mathbf{i} + \sin 2t\mathbf{j} + 3t\mathbf{k}$. We compute that

$$s = \int_{\pi}^{5\pi} |\mathbf{r}'(t)|\, dt = \int_{\pi}^{5\pi} \sqrt{5}\, dt = 4\pi\sqrt{5}.$$

13. (12 points)

(a) **(6 points)** Find a vector that is orthogonal to the vectors $\mathbf{i} + \mathbf{j} + \mathbf{k}$ and $2\mathbf{i} + 3\mathbf{j}$.

Solution. We compute that $\langle 1, 1, 1\rangle \times \langle 2, 3, 0\rangle = \langle -3, 2, 1\rangle$

(b) **(6 points)** Find the area of the parallelogram spanned by the vectors $\mathbf{i} + \mathbf{j} + \mathbf{k}$ and $2\mathbf{i} + 3\mathbf{j}$.

Solution. It is $|\langle -3, 2, 1\rangle| = \sqrt{14}$ square units.

14. (20 points)

(a) **(10 points)** Find an equation for a plane containing the points $(1, 0, 2)$, $(2, 3, 2)$, and $(1, 2, 5)$.

Solution. We compute the normal vector $\langle 1, 3, 0\rangle \times \langle 0, 2, 3\rangle = \langle 9, -3, 2\rangle$, from which we get the plane $9(x - 1) - 3(y - 0) + 2(z - 2) = 0$, or $9x - 3y + 2z = 13$.

(b) **(10 points)** Find an equation of the tangent plane at the point $(4, -2, 1)$ to the parametric surface S given by $\mathbf{r}(u, v) = v^2\mathbf{i} - uv\mathbf{j} + u^2\mathbf{k}$, where $0 \leq u \leq 3$ and $-3 \leq v \leq 3$.

Solution. We compute the normal vector $\langle 0, -v, 2u\rangle \times \langle 2v, -u, 0\rangle = \langle 2, 8, 8\rangle$, if $u = 1$, and $v = 2$. Hence, the tangent plane is given by the equation $2(x - 4) + 8(y + 2) + 8(z - 1) = 0$, or $x + 4y + 4z = 0$.

15. (12 points) Find the maximum and minimum values of the function $F(x, y) = xy$ subject to the constraint $x^2 + 3y^2 = 12$.

Solution. According to Lagrange's method, we solve the equation $\nabla f = \lambda \nabla g$, where λ is some positive constant. We have $y = 2\lambda x$ and $x = 8\lambda y$, from which we get that $\lambda = \pm\frac{1}{4}$. Then applying the $g(x, y) = x^2 + 4y^2 - 24$. We obtain $x = \pm\sqrt{12} = \pm 2\sqrt{3}$ and $y = \pm\sqrt{3}$. Substituting these coordinates into $F(x, y)$, we find that $F_{\text{max}} = F(2\sqrt{3}, \sqrt{3}) = 6 = F(-\pm 2\sqrt{3}, -\sqrt{3}) = 6$ and $F_{\text{min}} = F(-2\sqrt{3}, \sqrt{3}) = 6 = F(2\sqrt{3}, -\sqrt{3})$.