Solutions to Math 164 Midterm 2, Fall 2014.

1. a) The function is continuous since the denominator is nonzero at $(0,0)$. Thus the limit is simply the value at $(0,0)$ which is $\boxed{\frac{2}{\sqrt{4}} = 1}$.

   b) Along the path $x = t$, $y = 0$ the limit is $\lim_{t\to 0} \frac{0}{t^4} = 0$, but along the path $x = t$, $y = t^2$ the limit is $\lim_{t\to 0} \frac{2t^4}{2t^4} = 1$ so by the two-paths test the limit $\boxed{\text{does not exist}}$.

2. Solving for R gives $R = \left(R_1^{-1} + R_2^{-1} + R_3^{-1}\right)^{-1}$. Then $\frac{\partial R}{\partial R_2} = (-1) \cdot \left(R_1^{-1} + R_2^{-1} + R_3^{-1}\right)^{-2} \cdot (-1)R_2^{-2} = \frac{R^2}{R_2^2}$. When $R_1 = 30$ ohms, $R_2 = 45$ ohms, $R_3 = 90$ ohms, we get $R = 15$ ohms so $\frac{\partial R}{\partial R_2} = \frac{(15\text{ohm})^2}{(45\text{ohm})^2} = \boxed{\frac{1}{9}}$.

3. a) We have $f_x(x,y) = -y\,e^{-xy}$ and $f_y = -x\,e^{-xy}$, and these are both continuous functions since they are combinations of exponentials and polynomials (which are continuous).
   b) The linearization is $L(x,y) = f(1,0) + (x-1)\cdot f_x(1,0) + (y-0)\cdot f_y(1,0) = \boxed{1-y}$.
   c) We compute $L(1.03, 0.02) = 1 - 0.02 = \boxed{0.98}$.

4. Note that $\nabla f(x,y) = \langle x, y\rangle$ so $\nabla f(1,1) = \langle 1, 1\rangle$.

   a) The direction of maximum increase is in the direction of ∇f hence is $\boxed{\frac{\langle 1,1\rangle}{\sqrt{2}}}$.

   b) The direction of maximum increase is in the opposite direction of ∇f hence is $\boxed{\frac{\langle 1,1\rangle}{\sqrt{2}}}$.

   c) The function will have zero change in all directions orthogonal to the gradient. There are two such directions in the plane: $\boxed{\frac{\langle 1,-1\rangle}{\sqrt{2}}, \frac{\langle -1,1\rangle}{\sqrt{2}}}$ (they are both clearly orthogonal to ∇f).

5. We first find all critical points: we compute $f_x = 3x^2 - 3y$ and $f_y = 24y^2 - 3x$ and set them both equal to zero. The first equation gives $y = x^2$, and plugging in to the other equation gives $24x^4 - 3x = 0$ so $x = 0$ or $x = \frac{1}{2}$. There are two critical points, $(0,0)$ and $(\frac{1}{2}, \frac{1}{2})$.
   Then we compute $f_{xx} = 6x$, $f_{xy} = -3$, $f_{yy} = 48y$. So at $(0,0)$ we have $D = f_{xx}f_{yy} - (f_{xy})^2 = (0)(0) - (-3)^2 = -9$ so $\boxed{(0,0) \text{ is a saddle point}}$. At $(\frac{1}{2}, \frac{1}{2})$ we have $D = (3)(24) - (-3)^2 = 63 > 0$ and $f_{xx} = 3 > 0$, so $\boxed{(\frac{1}{2}, \frac{1}{2}) \text{ is a local minimum}}$.

6. For $f(x,y) = xy$ and $g(x,y) = \frac{x^2}{8} + \frac{y^2}{2} - 1$ we want to minimize/maximize f subject to $g(x,y) = 0$. We compute $\nabla f = \langle y, x\rangle$ and $\nabla g = \left\langle \frac{x}{4}, y\right\rangle$, so using the method of Lagrange multipliers yields the system $y = \lambda$, $\frac{x}{4}x = \lambda y$, $\frac{x^2}{8} + \frac{y^2}{2} - 1 = 0$. We have $x = \lambda y$ and then the first equation gives $y = \lambda\frac{\lambda y}{4}$ so $y = 0$, $\lambda = 2$, or $\lambda = -2$. If $y = 0$ then $x = 0$ also but this fails the third equation. So $\lambda = \pm 2$ and $x = \pm 2y$. In either case, the third equation gives $\frac{y^2}{2} + \frac{y^2}{2} - 1 = 0$ so $y^2 = 1$ and thus $y = \pm 1$. We get four candidate points $(2,1)$, $(2,-1)$, $(-2,1)$, and $(-2,-1)$. Since $f(2,1) = f(-2,-1) = 2$ and $f(-2,1) = f(2,-1) = -2$ the $\boxed{\text{maximum is } 2}$ and the $\boxed{\text{minimum is } -2}$.

7. $V = \int_0^1 \int_0^2 (10 + x^2 + 3y^2)\,dy\,dx = \int_0^1 \left(10y + x^2y + y^3\right)\Big|_{y=0}^{2} dx = \int_0^1 (28 + 2x^2)\,dx = \boxed{28 + \frac{2}{3} = \frac{86}{3}}$.