Math 164: Multivariable Calculus

Midterm Exam 1

October 18, 2012

NAME (please print legibly): ____

Your University ID Number: ____

CIRCLE YOUR INSTRUCTOR: Fili Madhu Mahmood

- **NO calculators, cell phones, iPods or other electronic devices are allowed during the exam.**
- **Show your work and justify your answers. You may not receive credit for a correct answer if insufficient work is shown or insufficient justification is given.**
- **There is no need to simplify your answers.**

QUESTION	VALUE	SCORE
1	10	
2	10	
3	15	
4	15	
5	10	
6	15	
7	10	
8	15	
Bonus	5	
TOTAL	100	

1. (10 points)

(a) Let $\vec{v} = \langle 1, 0, -2 \rangle$ and $\vec{w} = \langle 1, 2, 3 \rangle$. Find a **unit** vector orthogonal to both $\vec{v}$ and $\vec{w}$.
Solution: We note that the cross product of $\vec{v}$ and $\vec{w}$ is orthogonal to both.

$$\vec{v} \times \vec{w} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & -2 \\ 1 & 2 & 3 \end{vmatrix} = \langle 4, -5, 2 \rangle$$

The magnitude of $\langle 4, -5, 2 \rangle = \sqrt{16 + 25 + 4} = \sqrt{45}$.
So the two possible unit vectors orthogonal to both are $\pm\frac{1}{\sqrt{45}}\langle 4, -5, 2 \rangle$.

(b) Find the area of the triangle whose vertices are $P(3, 0, -6)$, $Q(-3, -6, -9)$ and the origin.
Solution: This area is $\frac{1}{2} \mid 3\vec{v} \times -3\vec{w} \mid = \frac{9}{2}\sqrt{45}$.

2. (10 points) Find the distance from the point $(1, 2, -3)$ to the plane whose equation is $z - 3 = 4(x - 2) + y$.

Solution:

The equation of the plane has general form $0 = 4x + y - z - 5$. So

$$d = \frac{|\ 4(1) + (2) - (-3) - 5\ |}{\sqrt{4^2 + 1^2 + 1^2}}$$

3. (15 points) A moving particle's position at any time t is given by

$$\vec{r}(t) = \langle 2\cos(2t), 3t, -2\sin(2t)\rangle.$$

(a) Calculate the velocity vector as a function of time t.

Solution: We differentiate each component of the position vector to get the velocity vector as a function of time t:

$$\vec{v}(t) = \vec{r}\,'(t) = \langle -4\sin(2t), 3, -4\cos(2t)\rangle.$$

(b) Find the arc length of the curve $\vec{r}(t)$ between $t = 0$ and $t = \pi$.

Solution: The arc length of the curve $\vec{r}(t)$ between $t = 0$ and $t = \pi$ is given by:

$$\begin{aligned} s &= \int_0^\pi |\vec{r}\,'(t)|\ dt \\ &= \int_0^\pi \sqrt{16\sin^2(2t) + 9 + 16\cos^2(2t)}\ dt \\ &= \int_0^\pi \sqrt{16+9}\ dt \qquad \text{since } \sin^2(2t) + \cos^2(2t) = 1 \\ &= \int_0^\pi 5\ dt \\ &= 5t\big|_0^\pi \\ &= 5\pi. \end{aligned}$$

(c) Reparametrize the curve in terms of arc length s measured from $t = 0$.

Solution: Based on the work in part (b), we know that the arc length s at any time t, measured from $t = 0$, is given by $s = \int_0^t |\vec{r}\,'(u)|\ du = 5t$. Therefore $t = \frac{s}{5}$. So

$$\vec{r}(s) = \left\langle 2\cos\left(\frac{2s}{5}\right), \frac{3s}{5}, -2\sin\left(\frac{2s}{5}\right)\right\rangle.$$

4. (15 points) Consider the function

$$f(x, y) = \frac{xy^2}{x^3 + 2y^3}.$$

(a) Does $f(x, y)$ approach a single value as (x, y) approaches $(0, 0)$ along the line $y = x$? If so, find this value.

Solution: Yes. Along $y = x$,

$$f(x, y) = f(x, x) = \frac{x \cdot x^2}{x^3 + 2x^3} = \frac{x^3}{3x^3} = \frac{1}{3},$$

for all $x \neq 0$. So as $(x, y) \to (0, 0)$ along $y = x$, $f(x, y) \to \frac{1}{3}$.

(b) Does $\lim_{(x,y)\to(0,0)} f(x, y)$ exist? If it exists, determine the value. If if doesn't exist, explain why.

Solution: No. Along the path $x = 0$,

$$f(x, y) = f(0, y) = \frac{0 \cdot y^2}{0^3 + 2y^3} = \frac{0}{2y^3} = 0,$$

for all $y \neq 0$. So as $(x, y) \to (0, 0)$ along $x = 0$, $f(x, y) \to 0$. This is not the same value as the one $f(x, y)$ approaches along $y = x$ in part (a). Therefore, $\lim_{(x,y)\to(0,0)} f(x, y)$ does not exist.

(c) Determine the set of points at which f is continuous.

Solution: Since f is a rational function, it is continuous wherever it is defined. Therefore, f is continuous on all points $(x, y) \in \mathbb{R}^2$ such that $x^3 + 2y^3 \neq 0$. That is, f is continuous on all points $(x, y) \in \mathbb{R}^2$ that do not lie on the line $x = -\sqrt[3]{2}y$.

5. (10 points) Consider the surface

$$\frac{x}{y} + \frac{y}{z} = 1.$$

We view z as an implicitly defined function of x, y.

(a) Find $\dfrac{\partial z}{\partial x}$. (Do not simplify.)

(b) Find $\dfrac{\partial z}{\partial y}$. (Do not simplify.)

6. (15 points)

Find **all** points on the ellipsoid

$$2x^2 + y^2 + 3z^2 = 9$$

for which the tangent plane to the surface at that point is parallel to the plane $2x + 2y - 3z = 0$.

7. (10 points)

For this problem, let $z = f(x, y) = x^2 e^{xy}$.

Give the unit vector that points in the direction of maximum increase along the surface when $(x, y) = (1, 1)$.

8. (15 points)

Consider the surface $z = f(x, y) = x^3 + y^3 - 3xy + 2$.

(a) Find all critical points of $f(x, y)$.

(b) Find all local extreme values of f and the points at which they occur. Find all saddle points.

Bonus Problem. (5 points) Prove that $\displaystyle\lim_{(x,y)\to(0,0)} \frac{2xy}{\sqrt{x^2+y^2}} = 0$.

Solution: Let $\epsilon > 0$. We want to show that there exists a $\delta > 0$ such that

$$\text{if } \sqrt{(x-0)^2+(y-0)^2} < \delta \text{ then } \left| \frac{2xy}{\sqrt{x^2+y^2}} - 0 \right| < \epsilon,$$

that is,

$$\text{if } \sqrt{x^2+y^2} < \delta \text{ then } \frac{2|x||y|}{\sqrt{x^2+y^2}} < \epsilon.$$

Let $\delta = \dfrac{\epsilon}{2}$. Assume $\sqrt{x^2+y^2} < \delta$. Since $y^2 \geq 0$, then

$$|x| = \sqrt{x^2} \leq \sqrt{x^2+y^2}.$$

Since $\sqrt{x^2+y^2} \geq 0$, then as long as x and y are not both 0, we have

$$\frac{|x|}{\sqrt{x^2+y^2}} \leq 1.$$

Multiplying both sides of this inequality by $2|y|$, we get

$$\begin{aligned}
\frac{2|x||y|}{\sqrt{x^2+y^2}} &\leq 2|y| \\
&\leq 2\sqrt{y^2} \\
&\leq 2\sqrt{x^2+y^2} && \text{since } x^2 \geq 0 \\
&< 2\delta && \text{by assumption} \\
&= 2 \cdot \frac{\epsilon}{2} \\
&= \epsilon.
\end{aligned}$$

Hence, by the formal definition of limit,

$$\lim_{(x,y)\to(0,0)} \frac{2xy}{\sqrt{x^2+y^2}} = 0.$$