MATH 164

Midterm 2 ANSWERS

November 21, 2011

1. (12 points) Consider the function $f(x,y) = x^2 - y^3 + 3y + 1$ for $(x,y) \in \mathbf{R}^2$.

(a) Find the critical points of f.

(b) Find the local maxima, minima, and saddle points of f.

(c) Are any of these points global maxima or minima?

Answer:

(a) We set the gradient of f equal to 0.

$$0 = \nabla f(x,y) = (2x, -3y^2 + 3) = (2x, -3(y-1)(y+1)).$$

The critical points of f are the solutions of this equation, namely $(0,1)$ and $(0,-1)$.

(b) We use the second derivative test. First we compute

$$D = \det \begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix} = \det \begin{pmatrix} 2 & 0 \\ 0 & -6y \end{pmatrix} = -12y$$

Thus at $(0,1)$ we have $D < 0$, so this is a saddle point.
At $(0,-1)$, $D > 0$ and $f_{xx} > 0$, so we have a local minimum.

(c) Since $f(x,y) \to +\infty$ along the line $y = 0$, $x \to +\infty$, and $f(x,y) \to -\infty$ along the line $x = 0, y \to +\infty$, it follows that none of these points are global maxima or minima.

2. (13 points) Find the extreme values of $f(x,y) = x^2 + xy + y^2 + x - y + 1$ subject to the constraint $g(x,y) = x^2 + y^2 = 1$. Go through the following steps:

(a) Use the Lagrange multiplier method to get two equations involving x, y and λ (the Lagrange multiplier).

(b) Starting from these two equations, show that $\lambda = \frac{3}{2}$ or $y = \frac{1}{1-2\lambda}$ and provide an argument to show $\lambda \neq \frac{1}{2}$.

(c) Find the four extreme points and evaluate f at these points to see which ones are maximum points and which ones are minimum points.

Answer:

(a) We need to use Lagrange multipliers. We will solve the system given by the equations:

$$\nabla f(x,y) = \lambda \nabla g(x,y)$$
$$g(x,y) = x^2 + y^2 = 1$$

From the first relation we get that

$$2x + y + 1 = 2x\lambda$$
$$2y + x - 1 = 2y\lambda$$

From the second equation, we get that $x = 1 + 2(\lambda - 1)y$; we plug in this expression for x into the first equation and we get

$$2(1 + 2(\lambda - 1)y)(1 - y) + y + 1 = 0$$

This is equivalent to

$$(2\lambda - 3)(y(-2\lambda + 1) - 1) = 0$$

(b) There are two possibilities: $\lambda = \frac{3}{2}$ or $y = \frac{1}{1-2\lambda}$, provided that $\lambda \neq \frac{1}{2}$ (note that λ couldn't be equal to $\frac{1}{2}$, since the equations would become $x + y = 1$ and $x + y = -1$, which is impossible).

(c) If $\lambda = \frac{3}{2}$ then $x - y = 1$ and plugging into the constraint, one obtains:

$$x^2 + (x-1)^2 = 1$$

or

$$2x(x-1) = 0$$

which has two solutions $x = 0$ and $x = 1$. Hence we get two points $(0,-1)$ and $(1,0)$. Notice that $f(0,-1) = f(1,0) = 3$.

If $y = \frac{1}{1-2\lambda}$, then $x = -\frac{1}{1-2\lambda} = -y$, so plugging into the constraint one obtains $2x^2 = 1$, so $x = \pm\frac{1}{\sqrt{2}}$. Hence we get two more points $\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$ and $\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$. Notice that $f\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right) = \frac{3+2\sqrt{2}}{2}$ and $f\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = \frac{3-2\sqrt{2}}{2}$.

We can conclude that $(0,-1)$ and $(1,0)$ are maximum points ($f_{\max} = 3$), while $\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ is the minimum point ($f_{\min} = \frac{3-2\sqrt{2}}{2}$).

3. (12 points) Consider the following integral:

$$I = \int_1^3 \int_1^2 xye^{xy^2}\, dxdy$$

Is it easier to solve this integral as written, or by changing the order of integration? Choose the easiest way and evaluate the integral.

Answer:

Integrating first over x would involve using integration by parts, while integrating first over y would only involve a u-substitution. So we make the following substitution and change the order (with x held constant for the first integral)

$$u = xy^2 \qquad du = 2xydy \qquad \frac{du}{2} = xydy$$

Then

$$\begin{aligned} I &= \int_1^2 \int_1^3 xye^{xy^2}\, dydx \\ &= \frac{1}{2}\int_1^2 \int_x^{9x} e^u du dx \\ &= \frac{1}{2}\int_1^2 e^u \Big|_x^{9x} dx \\ &= \frac{1}{2}\int_1^2 \left(e^{9x} - e^x\right) dx \\ &= \frac{1}{2}\left(\frac{1}{9}e^{9x} - e^x\right)\Bigg|_1^2 \\ &= \frac{e^{18}}{18} - \frac{e^9}{18} + \frac{e^2}{2} - \frac{e}{2} \end{aligned}$$

4. (13 points) Using polar coordinates, calculate the integral

$$\iint_D (x^2 + y^2)\; dx\; dy$$

where the domain D is inside the half-circle $x^2 + y^2 = 1$, $x \geq 0$, between the lines $y = \sqrt{3}x$, $x = \sqrt{3}y$. Go through the following steps:

(a) Determine the bounds for r and θ (it might be helpful to draw a sketch of the domain).

(b) Solve the new double integral.

Answer:

(a) Let $x = r\cos\theta$ and $y = r\sin\theta$. Since the domain is inside the circle, we know that $0 \leq r \leq 1$. The first line has slope $\tan\left(\frac{\pi}{3}\right)$, while the second has slope $\tan\left(\frac{\pi}{6}\right)$, which leads us to the bounds of θ: $\frac{\pi}{6} \leq \theta \leq \frac{\pi}{3}$.

(b) The integral becomes:

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \int_{0}^{1} r^2 r \; dr \; d\theta = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \int_{0}^{1} r^3 \; dr \; d\theta = \left(\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} d\theta \right) \cdot \left(\int_{0}^{1} r^3 \; dr \right) = \frac{\pi}{6} \cdot \frac{1}{4} = \frac{\pi}{24}$$

5. (13 points)

Find the center of mass of the region between the curve $y = x^2$, the x-axis, and the line $y = 1$.

(a) Assuming the density of region is 1, what is the mass of the region?

(b) Find the center of mass $\bar{x}$ in the x direction.

(c) Find the center of mass $\bar{y}$ in the y direction.

Answer:

Due to the typo noticed during the exam, three different solutions have been accepted. We present all three:

Solution 1

$0 \leq x \leq 1$ and $x^2 \leq y \leq 1$

(a) Since the density is 1, the mass of the region is the area. This can be done using a single integral,

$$m = \int_0^1 (1 - x^2) dx = 1 - \left.\frac{x^3}{3}\right|_0^1 = \frac{2}{3}$$

(b) We have

$$\begin{aligned}
\bar{x} &= \frac{1}{m}\int_0^1 \int_{x^2}^1 x dy dx \\
&= \frac{3}{2}\int_0^1 x \cdot (1-x^2) dx \\
&= \frac{3}{2} \cdot \left(\frac{x^2}{2} - \frac{x^4}{4}\right)\Big|_0^1 \\
&= \frac{3}{8}
\end{aligned}$$

(c) We have

$$\begin{aligned}
\bar{y} &= \frac{1}{m}\int_0^1 \int_{x^2}^1 y dy dx \\
&= \frac{3}{2}\int_0^1 \frac{y^2}{2}\Big|_{x^2}^1 dx \\
&= \frac{3}{4}\int_0^1 (1-x^4) dx \\
&= \frac{3}{4} \cdot \left(x - \frac{x^5}{5}\right)\Big|_0^1 \\
&= \frac{3}{5}
\end{aligned}$$

Solution 2

$-1 \leq x \leq 1$ and $x^2 \leq y \leq 1$

(a) Since the density is 1, the mass of the region is the area. This can be done using a single integral,

$$m = \int_{-1}^1 (1-x^2) dx = 1 - \frac{x^3}{3}\Big|_{-1}^1 = \frac{4}{3}$$

(b) We have

$$\begin{aligned}
\bar{x} &= \frac{1}{m}\int_{-1}^1 \int_{x^2}^1 x dy dx \\
&= \frac{3}{4}\int_{-1}^1 x \cdot (1-x^2) dx \\
&= \frac{3}{4} \cdot \left(\frac{x^2}{2} - \frac{x^4}{4}\right)\Big|_{-1}^1 \\
&= 0
\end{aligned}$$

(c) We have

$$\begin{aligned}
\bar{y} &= \frac{1}{m}\int_{-1}^{1}\int_{x^2}^{1} y dy dx \\
&= \frac{3}{4}\int_{-1}^{1} \left.\frac{y^2}{2}\right|_{x^2}^{1} dx \\
&= \frac{3}{8}\int_{-1}^{1}(1-x^4)dx \\
&= \frac{3}{8}\cdot\left.\left(x-\frac{x^5}{5}\right)\right|_{-1}^{1} \\
&= \frac{3}{8}
\end{aligned}$$

Solution 3

$0 \le x \le 1$ and $0 \le y \le x^2$

(a) Since the density is 1, the mass of the region is the area. This can be done using a single integral,

$$m = \int_0^1 x^2 dx = \left.\frac{x^3}{3}\right|_0^1 = \frac{1}{3}$$

(b) We have

$$\begin{aligned}
\bar{x} &= \frac{1}{m}\int_0^1\int_0^{x^2} x dy dx \\
&= 3\int_0^1 x\cdot x^2 dx \\
&= 3\cdot\left.\frac{x^4}{4}\right|_0^1 \\
&= \frac{3}{4}
\end{aligned}$$

(c) We have

$$\begin{aligned}
\bar{y} &= \frac{1}{m}\int_0^1\int_0^{x^2} y dy dx \\
&= 3\int_0^1 \left.\frac{y^2}{2}\right|_0^{x^2} dx \\
&= 3\int_0^1 \frac{x^4}{2} dx \\
&= 3\cdot\left.\frac{x^5}{5\cdot 2}\right|_0^1 \\
&= \frac{3}{10}
\end{aligned}$$

6. (12 points)

Calculate the following triple integral:

$$I = \iiint_E \frac{dx\ dy\ dz}{(1+x+y+z)^3}$$

where E is the domain bounded by the planes $x = 0$, $y = 0$, $z = 0$ and $x + y + z = 1$.

Answer:

The domain can be described as follows

$$E = \{(x, y, z) \in \mathbb{R} : 0 \le x \le 1, 0 \le y \le 1 - x, 0 \le z \le 1 - x - y\}.$$

The integral becomes:

$$I = \int_0^1\int_0^{1-x}\int_0^{1-x-y} \frac{1}{(1+x+y+z)^3}\ dz\ dy\ dx = \int_0^1\int_0^{1-x} \left. -\frac{1}{2}\frac{1}{(1+x+y+z)^2}\right|_{z=0}^{z=1-x-y} dy\ dx$$

$$= \int_0^1\int_0^{1-x} \left[-\frac{1}{8}+\frac{1}{2}\cdot\frac{1}{(1+x+y)^2}\right]\ dy\ dx = \int_0^1 \left.\left[-\frac{y}{8}-\frac{1}{2}\cdot\frac{1}{(1+x+y)}\right]\right|_{y=0}^{y=1-x} dx$$

$$= \int_0^1 \left[-\frac{3}{8}+\frac{x}{8}+\frac{1}{2}\frac{1}{1+x}\right]\ dx = \left.\left[-\frac{3x}{8}+\frac{x^2}{16}+\frac{1}{2}\ln(1+x)\right]\right|_0^1 = \frac{1}{2}\left(\ln 2 - \frac{5}{8}\right)$$

7. (12 points)

In this question we will set up an integral in cylindrical coordinates. **DO NOT EVALUATE THE INTEGRAL**.

The problem is to determine the volume of the region inside the sphere $x^2 + y^2 + z^2 = 2$ and above the plane $z = 1$.

(a) First find the value of r at the points of intersection of the sphere and the plane.

(b) Find the upper and lower limits of integration for z, as a function of r.

(c) Set up the integral, but **DO NOT EVALUATE IT**.

Answer:

(a) Using the fact that $r^2 = x^2 + y^2$, we can write the equations of the sphere as $r^2 + z^2 = 2$. Since $z = 1$ defines the plane, we can substitute into the equation for the sphere and get $r^2 + 1 = 2$ or $r^2 = 1$ or $r = 1$, since the radial coordinate r is never negative.

(b) The region inside the sphere $x^2 + y^2 + z^2 = 2$ and above the plane $z = 1$ gives us a lower limit of $z = 1$ and an upper limit of $z = \sqrt{2 - r^2}$, since $r^2 = x^2 + y^2$.

(c) Using cylindrical coordinates z, r, θ, we would use the differential $rdzdrd\theta$. Using the limits of integration from parts (a) and (b), we get

$$\text{Volume} = \int_0^{2\pi} \int_0^1 \int_1^{\sqrt{2-r^2}} rdzdrd\theta$$

8. (13 points) Using spherical coordinates, calculate the integral

$$I = \iiint_E \sqrt{x^2 + y^2 + z^2}\, dx\, dy\, dy$$

where $E = \{(x, y, z) \in \mathbb{R} : z \geq 0, x^2 + y^2 + z^2 \leq z\}$. Follow these steps:

(a) Knowing that $0 < \rho < g(\phi)$ and $0 \leq \phi \leq h(\pi)$, determine the expressions $g(\phi)$ and $h(\pi)$. Are there any restrictions for θ?

(b) Calculate I and show that $I < \frac{1}{3}$.

Answer:

(a) Let $x = \rho \sin\phi \cos\theta$, $y = \rho \sin\phi \sin\theta$ and $z = \rho \cos\phi$. We need to find the bounds for ρ, θ, ϕ.

From the description of the domain, we see that at the boundary: $x^2 + y^2 + z^2 = z$, which becomes $\rho^2 = \rho \cos\phi$, so $0 \leq \rho \leq \cos\phi$.

Another condition is give by $z \geq x^2 + y^2 + z^2 \geq 0$, which means the domain lies above the (xy)-plane, so ϕ is bounded by: $0 < \phi \leq \frac{\pi}{2}$. For θ there isn't any restriction, so $0 \leq \theta \leq 2\pi$.

(b) The integral becomes:

$$
\begin{aligned}
I &= \int_0^{2\pi} \int_0^{\frac{\pi}{2}} \int_0^{\cos\phi} \rho \cdot \rho^2 \sin\phi \; d\rho \; d\phi \; d\theta = \int_0^{2\pi} \int_0^{\frac{\pi}{2}} \left[\frac{\rho^4}{4}\right] \Big|_0^{\cos\phi} \sin\phi \; d\phi \; d\theta \\
&= \int_0^{2\pi} \int_0^{\frac{\pi}{2}} \frac{\cos^4\phi}{4} \sin\phi \; d\phi \; d\theta = \left[\int_0^{2\pi} d\theta\right] \left[\int_0^{\frac{\pi}{2}} \frac{\cos^4\phi}{4} \sin\phi \; d\phi\right] = 2\pi \cdot \frac{1}{4} \cdot \left(-\frac{\cos^5\phi}{5}\Big|_0^{\frac{\pi}{2}}\right) \\
&= \frac{\pi}{10} < \frac{1}{3}.
\end{aligned}
$$