MATH 164

Final ANSWERS

December 21, 2011

Part A

1. (9 points) Find an equation for the plane passing through the points

$$(1,2,3) \qquad (1,0,-2) \qquad (0,-2,1)$$

Answer:

Call the points p, q, r respectively. We need to find 2 vectors parallel to the plane. Subtacting, we get

$$\begin{aligned} u &= p - q = (0,2,5) \\ v &= p - r = (1,4,2) \end{aligned}$$

Then $n = u \times v$ is perpendicular to the plane. We get

$$\begin{aligned} n = u \times v &= \det \begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 2 & 5 \\ 1 & 4 & 2 \end{pmatrix} \\ &= (4-20)\mathbf{i} + 5\mathbf{j} - 2\mathbf{k} \\ &= -16\mathbf{i} + 5\mathbf{j} - 2\mathbf{k} \end{aligned}$$

The equation of the plane becomes

$$\begin{aligned} 0 &= \big((x,y,z) - p\big) \cdot n \\ &= (x-1, y-2, z-3) \cdot (-16, 5, -2) \\ &= -16x + 5y - 2z + 12 \end{aligned}$$

Note that other choices of vectors to subtract could result in an equation which looks differently on the surface.

2. (8 points) Calculate the following double integral:

$$\iint_D \sqrt{1 - \frac{x^2}{4} - \frac{y^2}{9}} \, dA$$

where $D = \left\{(x, y) \in \mathbb{R}^2 \middle| \frac{x^2}{4} + \frac{y^2}{9} \leq 1\right\}$ (the interior of an ellipse).

Follow these steps:

(a) Use the substitution $x = 2r\cos\theta$ and $y = 3r\sin\theta$. Write the domain E of r and θ and calculate the Jacobian $J(r, \theta) = \frac{\partial(x,y)}{\partial(r,\theta)}$ of this change of variables.

(b) Now calculate

$$\iint\limits_D f(x, y)\, dA = \iint\limits_D \sqrt{1 - \frac{x^2}{4} - \frac{y^2}{9}}\, dA = \iint\limits_E f(x(r, \theta), y(r, \theta))\, |J(r, \theta)|\, dr\, d\theta.$$

Answer:

(a) The domain $E = [0, 1] \times [0, 2\pi]$ and the Jacobian is $\frac{\partial(x,y)}{\partial(r,\theta)} = 6r$.

(b)

$$\begin{aligned}\iint\limits_D \sqrt{1 - \frac{x^2}{4} - \frac{y^2}{9}}\, dA &= \iint\limits_E \sqrt{1 - r^2}\, |J(r, \theta)|\, dr\, d\theta = \int_0^{2\pi}\int_0^1 \sqrt{1 - r^2} \cdot 6r\, dr\, d\theta \\ &= \left[\int_0^{2\pi} d\theta\right]\left[\int_0^1 6r\sqrt{1 - r^2}\, dr\right] = 12\pi \cdot \frac{-1}{3}(1 - r^2)^{\frac{3}{2}}\Big|_0^1 \\ &= 4\pi.\end{aligned}$$

□

3. (9 points) Suppose that

$$\frac{\partial f}{\partial x}(1, 2) = 7$$
$$\frac{\partial f}{\partial y}(1, 2) = -3$$

and

$$x(t) = t^2$$
$$y(t) = 2t^3$$

Find

$$\frac{d}{dt} f(x(t), y(t))\Big|_{t=1}$$

Answer:

The chain rule says that

$$\frac{d}{dt} f(x(t), y(t)) = \frac{\partial f}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dt}$$

Now $x(1) = 1$ and $y(1) = 2$, and

$$\begin{array}{ll} \frac{dx}{dt} = 2t & \frac{dx}{dt}(1) = 2 \\ \frac{dy}{dt} = 6t^2 & \frac{dy}{dt}(1) = 6 \end{array}$$

so

$$\frac{d}{dt} f(x(t), y(t)) = 7 \cdot 2 + (-3) \cdot 6 = -4$$

4. (8 points)

A solid is bounded by the four planes given below. Its density is given by the function $f(x, y, z) = \sin(x + y + z)$. Calculate the mass of the solid.

(1) $x = 0$

(2) $y = 0$

(3) $z = 0$

(4) $x + y + z = \frac{\pi}{2}$

Answer:

One needs to calculate the triple integral

$$\iiint_E \sin(x + y + z)\; dV$$

where E is bounded by the planes $x = 0, y = 0, z = 0$ and $x + y + z = \frac{\pi}{2}$. This becomes:

$$\iiint_E \sin(x+y+z)\, dV = \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}-x} \int_0^{\frac{\pi}{2}-x-y} \sin(x+y+z)\, dz\, dy\, dx$$
$$= \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}-x} -\cos(x+y+z)\Big|_0^{\frac{\pi}{2}-x-y} dy\, dx = \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}-x} \cos(x+y)\, dy\, dx$$
$$= \int_0^{\frac{\pi}{2}} \sin(x+y)\Big|_0^{\frac{\pi}{2}-x} dx = \int_0^{\frac{\pi}{2}} (1-\sin x)\, dx = \frac{\pi}{2} + \cos x\Big|_0^{\frac{\pi}{2}} = \frac{\pi}{2} - 1.$$

So the mass is $\frac{\pi}{2} - 1$. □

5. (8 points) Consider the function

$$f(x,y) = x^2 - 3\frac{x}{y} + y^3$$

Find a vector v which is tangent to the level curve of the surface $z = f(x,y)$ at the point $(x,y) = (2,-1)$.

Answer:

Recall that the gradient vector ∇f points in the direction of greatest increase of f. We have

$$\nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j}$$
$$= \left(2x - \frac{3}{y}\right)\mathbf{i} + \left(3\frac{x}{y^2} + 3y^2\right)\mathbf{j}$$

and

$$\nabla f(2,-1) = 7\mathbf{i} + 9\mathbf{j}$$

This is the direction of greatest increase of f, which is perpendicular to the level surfaces. So we need to find a vector perpendicular to $\nabla f(2,-1) = 7\mathbf{i} + 9\mathbf{j}$. Such a vector could be

$$9\mathbf{i} - 7\mathbf{j}$$

or any nonzero multiple of it.

6. (8 points)

Consider a particle whose position is given by the curve $\mathbf{r}(t) = e^t \cos t \cdot \mathbf{i} + e^t \sin t \cdot \mathbf{j} + te^t \cdot \mathbf{k}$, for $t \in [0,1]$.

(a) Calculate its velocity, acceleration and speed.

(b) Write the integral which gives the total distance traveled by the particle. **Do not evaluate it**.

Answer:

(a)

$$\begin{aligned}
\mathbf{v}(t) &= \mathbf{r}'(t) = e^t(\cos t - \sin t)\cdot\mathbf{i} + e^t(\sin t + \cos t)\cdot\mathbf{j} + e^t(1+t)\cdot\mathbf{k} \\
\mathbf{a}(t) &= \mathbf{v}'(t) = -2e^t\sin t\cdot\mathbf{i} + 2e^t\cos t\cdot\mathbf{j} + e^t(2+t)\cdot\mathbf{k} \\
v(t) &= |\mathbf{v}(t)| = |\mathbf{r}'(t)| = e^t\sqrt{t^2+2t+3}
\end{aligned}$$

(b)

$$\text{dist} = \int_0^1 |\mathbf{r}'(t)|\, dt = \int_0^1 e^t\sqrt{t^2+2t+3}\, dt$$

Part B

7. (8 points) Suppose that

$$\begin{aligned}
u &= xe^{xy} \\
v &= x^2+y^2
\end{aligned}$$

Let D, E be regions in $\mathbf{R}^2$ such that if we change variables from (x,y) to (u,v) and $x,y \in D$, then $(u,v)\in E$. Furthermore, the map between the two regions is one to one. Then we can write

$$\iint_E f(u,v)dudv = \iint_D f(xe^{xy}, x^2+y^2)h(x,y)dxdy.$$

Find $h(x,y)$.

Answer:

According to the change of variable formula, $h(x,y)$ must be the Jacobian determinant

$$\begin{aligned}
h(x,y) = \frac{\partial(u,v)}{\partial(x,y)} &= \left|\det\begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y}\end{pmatrix}\right| \\
&= \left|\det\begin{pmatrix} e^{xy}+xye^{xy} & x^2e^{xy} \\ 2x & 2y\end{pmatrix}\right| \\
&= |2e^{xy}(y+xy^2-x^3)|
\end{aligned}$$

8. (8 points)

Let $\mathbf{F}$ be a vector field defined as follows:

$$\mathbf{F} = x^3\ \mathbf{i} + y^3\ \mathbf{j} + z^3\ \mathbf{k}$$

and S be the sphere given by the equation $x^2 + y^2 + z^2 = 4$.

(a) Calculate div $\mathbf{F}$.

(b) Write down a suitable parametrization of the surface S. In this parametrization, write the expression of $\iint_S \mathbf{F} \cdot\ d\mathbf{S}$.
Do not evaluate it yet.

(c) State the divergence theorem in this setting, decide if it is useful for evaluating

$$\iint_S \mathbf{F} \cdot\ d\mathbf{S}.$$

Finish this exercise by calculating the above integral.

Answer:

(a) div $\mathbf{F} = 3x^2 + 3y^2 + 3z^2$.

(b) Since it is a sphere of radius 2, a suitable parametrization will be given by $x = 2\sin\phi\cos\theta$, $y = 2\sin\phi\sin\theta$ and $z = 2\cos\phi$, with $\phi \in [0, \pi]$ and $\theta \in [0, 2\pi]$. Then

$$\begin{aligned}
\mathbf{r}_\phi &= 2\cos\phi\cos\theta\ \mathbf{i} + 2\cos\phi\sin\theta\ \mathbf{j} - 2\sin\phi\ \mathbf{k} \\
\mathbf{r}_\theta &= -2\sin\phi\sin\theta\ \mathbf{i} + 2\sin\phi\cos\theta\ \mathbf{j} \\
\mathbf{r}_\phi \times \mathbf{r}_\theta &= 4\sin^2\phi\cos\theta\ \mathbf{i} + 4\sin^2\phi\sin\theta\ \mathbf{j} + 4\sin\phi\cos\phi\ \mathbf{k}
\end{aligned}$$

Then

$$\begin{aligned}
\iint_S \mathbf{F} \cdot\ d\mathbf{S} &= \int_0^{2\pi}\int_0^{\pi} \mathbf{F} \cdot (\mathbf{r}_\phi \times \mathbf{r}_\theta)\ d\phi\ d\theta \\
&= \int_0^{2\pi}\int_0^{\pi} 32(\sin^5\phi\cos^4\theta + \sin^5\phi\sin^4\theta + \sin\phi\cos^4\phi)\ d\phi\ d\theta
\end{aligned}$$

(c) The divergence theorem says that

$$\iint_S \mathbf{F} \cdot \, d\mathbf{S} = \iiint_E \text{div } \mathbf{F} \, dV$$

where E is the solid region enclosed by the surface S.

It is easier to actually calculate the corresponding triple integral:

$$\iiint_E 3(x^2 + y^2 + z^2) \, dV$$

We will use spherical coordinates, since E is the interior of the sphere of radius 2.

$$\begin{aligned}
\iiint_E 3(x^2 + y^2 + z^2) \, dV &= \int_0^{2\pi} \int_0^{\pi} \int_0^{2} 3\rho^2 \, \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta \\
&= \left[\int_0^{2\pi} d\theta \right] \left[\int_0^{\pi} \sin\phi \, d\phi \right] \left[\int_0^{2} 3\rho^4 \, d\rho \right] \\
&= 2\pi \cdot (-\cos(\phi))\Big|_0^{\pi} \cdot \left(3 \cdot \frac{\rho^5}{5}\right)\Big|_0^{2} \\
&= 2\pi \cdot 2 \cdot \frac{96}{5} = \frac{384\pi}{5}
\end{aligned}$$

□

9. (9 points) Suppose that

$$\mathbf{F}(x, y) = (\sin x \cos y, x^2 y)$$

Let D be the interior of the unit square with vertices at $(0,0), (0,1), (1,0), (1,1)$.

(a) Write Green's theorem for this vector field and this region. Find specific integrals for both the line integral and the double integral. **Do not evaluate**.

(b) Evaluate the line integral from part (a).

Answer:

(a) Green's theorem would say

$$\int_C P dx + Q dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right).$$

Starting with the right hand side, and using the fact that D is a square as given in the problem, we get

$$\iint\limits_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) = \int_0^1 \int_0^1 (2xy + \sin x \sin y)\, dxdy$$

For the left hand side, we have to break the boundary of the square into 4 line segments and integrate in the counterclockwise direction. We get

$$\begin{aligned}\int_C Pdx + Qdy &= \int_0^1 \sin x \cos(0) dx + \int_0^1 ydy - \int_0^1 \sin x \cos(1) dx - \int_0^1 0 \cdot dy \\ &= \int_0^1 \sin x dx + \int_0^1 ydy - \int_0^1 \sin x \cos(1) dx\end{aligned}$$

(b) Evaluating the preceding integral, we get

$$\begin{aligned}\int_C Pdx + Qdy &= -\cos x\Big|_0^1 + \frac{y^2}{2}\Big|_0^1 + \cos x \cos(1)\Big|_0^1 \\ &= 1 - \cos(1) + \frac{1}{2} + \cos^2(1) - \cos(1) \\ &= \frac{3}{2} - 2\cos(1) + \cos^2(1).\end{aligned}$$

10. (8 points)

Calculate the area of the surface S given by the equation $z = \frac{1}{2}xy$, which lies inside the cylinder $x^2 + y^2 = 4$. Follow the steps:

(a) Write a suitable parametrization of the surface, calculate the vector $\mathbf{r}_x \times \mathbf{r}_y$ and its length.

(b) Determine the domain D of the parameters, i.e. determine what is the projection of the surface onto the (xy)-plane and show that $\text{Area}(S) = \iint\limits_D \frac{1}{2}\sqrt{4 + x^2 + y^2}\; dx\; dy$.

(c) Calculate the integral from above. **Hint:** you may want to change coordinates.

Answer:

(a) Since the surface is given explicitly, a suitable parametrization is $\mathbf{r}(x,y) = x\mathbf{i}+y\mathbf{j}+\frac{xy}{2}\mathbf{k}$. Then we have:

$$\begin{aligned}
\mathbf{r}_x &= \mathbf{i} + \frac{y}{2}\mathbf{k} \\
\mathbf{r}_y &= \mathbf{j} + \frac{x}{2}\mathbf{k} \\
\mathbf{r}_x \times \mathbf{r}_y &= -\frac{y}{2}\mathbf{i} - \frac{x}{2}\mathbf{j} + \mathbf{k} \\
|\mathbf{r}_x \times \mathbf{r}_y| &= \sqrt{1 + \frac{x^2}{4} + \frac{y^2}{4}}
\end{aligned}$$

(b) The domain is the disk $D = \{(x,y) \in \mathbb{R}^2 | x^2 + y^2 \le 4\}$, which is obtained by projecting the surface (which lies inside the cylinder $x^2 + y^2 = 4$) onto the (xy)-plane.

Hence the area of the surface is:

$$\begin{aligned}
\text{Area}(S) &= \iint_S 1 \cdot \, dS = \iint_D 1 \cdot |\mathbf{r}_x \times \mathbf{r}_y| \, dA = \iint_D \sqrt{1 + \frac{x^2}{4} + \frac{y^2}{4}} \, dA \\
&= \iint_D \frac{1}{2}\sqrt{4 + x^2 + y^2} \, dx \, dy
\end{aligned}$$

(c) The easiest way to calculate the integral is to use polar coordinates. Then it becomes:

$$\begin{aligned}
\text{Area}(S) &= \iint_D \frac{1}{2}\sqrt{4 + x^2 + y^2} \, dx \, dy \\
&= \frac{1}{2}\int_0^{2\pi}\int_0^2 \sqrt{4 + r^2} \cdot r \, dr \, d\theta = \frac{1}{2}\left[\int_0^{2\pi} d\theta\right]\left[\int_0^2 r\sqrt{4 + r^2} \, dr\right] \\
&= \frac{1}{2} \cdot 2\pi \cdot \left[\frac{1}{3}(4 + r^2)^{\frac{3}{2}}\right]\Big|_0^2 = \frac{8\pi(2\sqrt{2} - 1)}{3}
\end{aligned}$$

□

11. (9 points) Suppose that S is the surface defined by $z = 1 - x^2 - y^2$ for $z \ge 0$, and C is the boundary of this surface. Let

$$\mathbf{F}(x,y,z) = x\mathbf{i} + z\mathbf{j} - 3y\mathbf{k}.$$

(a) What does Stokes theorem say about this situation? First write down what Stokes theorem says in general, using the symbols S, C, $\mathbf{F}$.

(b) Evaluate the line integral which occurs in Stokes theorem.

(c) Find curl($\mathbf{F}$).

(d) Let r, θ be polar coordinates in the x-y plane. Let $R(r, \theta)$ be a parameterization of the surface. Find R_r, R_θ, and a normal vector to the surface (not necessarily a unit vector).

Answer:

(a) Stokes theorem says that

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_D \mathrm{curl}(\mathbf{F}) \cdot d\mathbf{S}$$

(b) Looking at the left hand side, we see that C is the unit circle in the x-y plane, so that

$$\begin{aligned} \mathbf{r}(t) &= \cos t\mathbf{i} + \sin t\mathbf{j} \\ \mathbf{r}'(t) &= -\sin t\mathbf{i} + \cos t\mathbf{j} \\ \mathbf{F}(\mathbf{r}(t)) &= \cos t\mathbf{i} - 3\sin t\mathbf{k} \\ \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) &= -\cos t \sin t \end{aligned}$$

Evaluating this integral, which also answers part (b), gives

$$\int_C \mathbf{F} \cdot d\mathbf{r} = -\int_0^{2\pi} \cos t \sin t dt = -\frac{1}{2} \sin^2 t \Big|_0^{2\pi} = 0$$

(c)

$$\mathrm{curl}(\mathbf{F}) = \det \begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & z & -3y \end{pmatrix} = -4\mathbf{i}$$

(d) Using polar coordinates in the x-y plane as suggested, we find

$$\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \\ z &= 1 - x^2 - y^2 = 1 - r^2 \end{aligned}$$

Thus,

$$R(r, \theta) = r\cos\theta\mathbf{i} + r\sin\theta\mathbf{j} + (1 - r^2)\mathbf{k}$$

and

$$R_r = \cos\theta\mathbf{i} + \sin\theta\mathbf{j} - 2r\mathbf{k}$$
$$R_\theta = -r\sin\theta\mathbf{i} + r\cos\theta\mathbf{j}$$

To find a normal vector to the surface, we take the cross product

$$\begin{aligned} R_r \times R_\theta &= \det\begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos\theta & \sin\theta & -2r \\ -r\sin\theta & r\cos\theta & 0 \end{pmatrix} \\ &= 2r^2\cos\theta\mathbf{i} + 2r^2\sin\theta\mathbf{j} + (r\cos^2\theta + r\sin^2\theta)\mathbf{k} \\ &= 2r^2\cos\theta\mathbf{i} + 2r^2\sin\theta\mathbf{j} + r\mathbf{k} \end{aligned}$$

12. (8 points)

Show that the vector field $\mathbf{F} = (y+z)\ \mathbf{i} + (x+z)\ \mathbf{j} + (x+y)\ \mathbf{k}$ is conservative and find the function f such that $\mathbf{F} = \nabla f$. Calculate $\int_C \mathbf{F}\cdot\ d\mathbf{s}$ where C is a simple curve connecting the points $P_1 = (0,1,2)$ and $P_2 = (1,2,3)$.

Answer:

One notices that curl $\mathbf{F} = \nabla\times\mathbf{F} = \mathbf{0}$, so the field is conservative. Hence there is a function f such that $\mathbf{F} = \nabla f$.

$$\begin{aligned} \frac{\partial f}{\partial x} &= y + z \Rightarrow f(x,y,z) = xy + xz + C(y,z) + c \\ \frac{\partial f}{\partial y} &= x + z \Rightarrow f(x,y,z) = xy + yz + C(x,z) + c \\ \frac{\partial f}{\partial z} &= x + y \Rightarrow f(x,y,z) = xz + yz + C(x,y) + c \end{aligned}$$

The only solution is $f(x,y,z) = xy + yz + xz + c$, where c is a constant.

By applying the fundamental theorem for line integrals, one has:

$$\int_C \mathbf{F}\cdot\ d\mathbf{s} = \int_C \nabla f\cdot\ d\mathbf{s} = f(1,2,3) - f(0,1,2) = 2 + 6 + 3 - 2 = 9.$$