# Math 164: Multi-Dimensional Calculus

Midterm 1

October 21, 2008

**NAME (please print legibly):** ______________________________

**Your University ID Number:** ______________________________

**Indicate your instructor with a check in the box:**

| | | |
|---|---|---|
| Nicholas Rogers | MWF 10:00 - 10:50 AM | |
| Sema Salur | MWF 9:00 - 9:50 AM | |

- **The presence of calculators, cell phones, iPods and other electronic devices at this exam is strictly forbidden.**
- **Show your work and justify your answers. You may not receive full credit for a correct answer if insufficient work is shown or insufficient justification is given.**
- **Clearly circle or label your simplified final answers.**
- **You are responsible for checking that this exam has all 10 pages.**

| QUESTION | VALUE | SCORE |
|---|---|---|
| 1 | 12 | |
| 2 | 12 | |
| 3 | 12 | |
| 4 | 10 | |
| 5 | 12 | |
| 6 | 16 | |
| 7 | 8 | |
| 8 | 10 | |
| 9 | 8 | |
| TOTAL | 100 | |

**Formulas**

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

$$\vec{u} \cdot \vec{v} = u_1 v_1 + u_2 v_2 + u_3 v_3 \qquad \vec{u} \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

$$\vec{u} \cdot \vec{v} = |\vec{u}||\vec{v}| \cos\theta \qquad |\vec{u} \times \vec{v}| = |\vec{u}||\vec{v}| \sin\theta$$

$$\text{comp}_{\vec{u}}\vec{v} = \frac{\vec{u} \cdot \vec{v}}{|\vec{u}|} \qquad \text{proj}_{\vec{u}}\vec{v} = \frac{\vec{u} \cdot \vec{v}}{|\vec{u}|^2}\vec{u}$$

$$\vec{r} = \vec{r_0} + t\vec{v} \qquad \vec{n} \cdot (\vec{r} - \vec{r_0}) = 0$$

$$L = \int_a^b |\vec{r}\,'(t)|\, dt$$

$$\vec{T}(t) = \frac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|} \qquad \vec{N}(t) = \frac{\vec{T}'(t)}{|\vec{T}'(t)|} \qquad \vec{B}(t) = \vec{T}(t) \times \vec{N}(t)$$

$$\kappa(t) = \frac{|\vec{T}'(t)|}{|\vec{r}\,'(t)|} = \frac{|\vec{r}\,'(t) \times \vec{r}\,''(t)|}{|\vec{r}\,'(t)|^3}$$

$$L(x, y) = f(x_0, y_0) + \frac{\partial f}{\partial x}(x_0, y_0) \cdot (x - x_0) + \frac{\partial f}{\partial y}(x_0, y_0) \cdot (y - y_0)$$

**1. (12 points)** Let $\vec{v} = 4\vec{i} - \vec{j} + \vec{k}$ and $\vec{w} = 2\vec{i} + 3\vec{j} - \vec{k}$. Find:

(a) $\vec{v} \cdot \vec{w}$

$$\vec{v} \cdot \vec{w} = 4 \cdot 2 - 1 \cdot 3 + 1(-1) = \boxed{4.}$$

(b) $\cos\theta$, where $\theta$ is the angle between $\vec{v}$ and $\vec{w}$.

Since $\vec{v} \cdot \vec{w} = |\vec{v}||\vec{w}| \cos\theta$,

$$\cos\theta = \frac{\vec{v} \cdot \vec{w}}{|\vec{v}||\vec{w}|} = \frac{4}{\sqrt{4^2 + (-1)^2 + 1^2}\sqrt{2^2 + 3^2 + (-1)^2}} = \frac{4}{\sqrt{18}\sqrt{14}} = \boxed{\frac{2}{3\sqrt{7}}.}$$

(c) a scalar $s$ such that $\vec{v}$ is orthogonal to $\vec{v} - s\vec{w}$.

We'd like $\vec{v} \cdot (\vec{v} - s\vec{w}) = 0$.

$$\begin{aligned} \vec{v} \cdot (\vec{v} - s\vec{w}) &= \vec{v} \cdot \vec{v} - s(\vec{v} \cdot \vec{w}) \\ &= |\vec{v}|^2 - 4s = 18 - 4s. \end{aligned}$$

Thus $4s = 18$, or $\boxed{s = \frac{9}{2}.}$

**2. (12 points)** Find an equation for the plane that passes through the origin and is parallel to the vectors $\vec{v} = \vec{i} - 2\vec{j} - 3\vec{k}$ and $\vec{w} = -\vec{i} + \vec{j} + 2\vec{k}$.

The normal vector for this plane must be perpendicular to both of the given vectors, so we use the cross product.

$$\begin{aligned} \vec{v} \times \vec{w} &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & -2 & -3 \\ -1 & 1 & 2 \end{vmatrix} \\ &= \vec{i} \begin{vmatrix} -2 & -3 \\ 1 & 2 \end{vmatrix} - \vec{j} \begin{vmatrix} 1 & -3 \\ -1 & 2 \end{vmatrix} + \vec{k} \begin{vmatrix} 1 & -2 \\ -1 & 1 \end{vmatrix} \\ &= -\vec{i} + \vec{j} - \vec{k}. \end{aligned}$$

Then the equation for the plane is given by $\vec{n} \cdot (\vec{r} - \vec{r_0}) = 0$. Since $\vec{r_0} = \langle 0, 0, 0 \rangle$ is the origin, $\vec{n} \cdot \vec{r_0} = 0$ and we obtain

$$\boxed{-x + y - z = 0.}$$

**3. (12 points)** A triangle in $\mathbb{R}^3$ has vertices $A(3,4,-1)$, $B(0,0,3)$ and $C(1,0,-4)$.

(a) Find the perimeter of the triangle.

We use the distance formula three times:

$$\begin{aligned} AB &= \sqrt{(0-3)^2+(0-4)^2+(3-(-1))^2} = \sqrt{(-3)^2+(-4)^2+4^2} = \sqrt{41}; \\ AC &= \sqrt{(1-3)^2+(0-4)^2+(-4-(-1))^2} = \sqrt{(-2)^2+(-4)^2+(-3)^2} = \sqrt{29}; \\ BC &= \sqrt{(1-0)^2+(0-0)^2+(-4-3)^2} = \sqrt{1^2+0^2+7^2} = \sqrt{50} = 5\sqrt{2}. \end{aligned}$$

Thus the total perimeter is $\boxed{\sqrt{41}+\sqrt{29}+5\sqrt{2}.}$

(b) Find the area of the triangle.

Recall that the magnitude of the cross product $\overrightarrow{AB}\times\overrightarrow{AC}$ is the area of the parallelogram spanned by $\overrightarrow{AB}$ and $\overrightarrow{AC}$. The area of the triangle is half the area of this parallelogram. Now $\overrightarrow{AB} = -3\vec{i}-4\vec{j}+4\vec{k}$ and $\vec{AC} = -2\vec{i}-4\vec{j}-3\vec{k}$, so

$$\begin{aligned} \overrightarrow{AB}\times\overrightarrow{AC} &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -3 & -4 & 4 \\ -2 & -4 & -3 \end{vmatrix} \\ &= \vec{i}\begin{vmatrix} -4 & 4 \\ -4 & -3 \end{vmatrix} - \vec{j}\begin{vmatrix} -3 & 4 \\ -2 & -3 \end{vmatrix} + \vec{k}\begin{vmatrix} -3 & -4 \\ -2 & -4 \end{vmatrix} \\ &= 28\vec{i}-17\vec{j}+4\vec{k}. \end{aligned}$$

So the area of the triangle is

$$A = \tfrac{1}{2}|\overrightarrow{AB}\times\overrightarrow{AC}| = \frac{\sqrt{28^2+(-17)^2+4^2}}{2} = \frac{\sqrt{784+289+16}}{2} = \frac{\sqrt{1089}}{2} = \boxed{\frac{33}{2}.}$$

**4. (10 points)** Find the parametric equations for the tangent line to the graph of the vector function $\vec{F}(t) = \langle \sin t, \cos t, 3t \rangle$ at the point $P_0$ corresponding to $t_0 = 0$.

To find the tangent line, we need to know $P_0$ and the tangent vector $\vec{F}\,'(0)$.

$$\begin{aligned} P_0 &= \vec{F}(0) = \langle \sin 0, \cos 0, 3(0) \rangle = \langle 0, 1, 0 \rangle. \\ \vec{F}\,'(t) &= \langle \cos t, -\sin t, 3 \rangle; \\ \vec{F}\,'(0) &= \langle \cos 0, -\sin 0, 3 \rangle = \langle 1, 0, 3 \rangle. \end{aligned}$$

So the parametrization of the tangent line is given by

$$\vec{r}(t) = P_0 + t\vec{F}\,'(0) = \langle 0, 1, 0 \rangle + t\langle 1, 0, 3 \rangle,$$

or equivalently,

$$\boxed{x(t) = t; \qquad y(t) = 1; \qquad z(t) = 3t.}$$

**5. (12 points)** Find the velocity $\vec{v}(t)$, the speed $|\vec{v}(t)|$ and the acceleration $\vec{a}(t)$ for the body with position vector $\vec{r}(t) = t\vec{i} + 2t\vec{j} + te^t\vec{k}$.

$$\begin{aligned} \vec{v}(t) &= \vec{r}\,'(t) = \boxed{\vec{i} + 2\vec{j} + (e^t + te^t)\vec{k}.} \\ |\vec{v}(t)| &= \sqrt{1^2 + 2^2 + (e^t + te^t)^2} = \boxed{\sqrt{5 + e^{2t}(t+1)^2}.} \\ \vec{a}(t) &= \vec{v}\,'(t) = (e^t + e^t + te^t)\vec{k} = \boxed{e^t(t+2)\vec{k}.} \end{aligned}$$

**6. (16 points)** For the curve given by $\vec{r}(t) = (\sin t)\vec{i} + (\cos t)\vec{j} + t\vec{k}$, find:

(a) a <u>unit</u> tangent vector $\vec{T}$ at the point on the curve where $t = \pi$.

The vector $\vec{r}\,'(t) = (\cos t)\vec{i} - (\sin t)\vec{j} + \vec{k}$ is a tangent vector, so a unit tangent vector is given by

$$\vec{T}(t) = \frac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|} = \frac{(\cos t)\vec{i} - (\sin t)\vec{j} + \vec{k}}{\sqrt{\cos^2 t + \sin^2 t + 1}} = \frac{(\cos t)\vec{i} - (\sin t)\vec{j} + \vec{k}}{\sqrt{2}}.$$

$$\vec{T}(\pi) = \frac{(\cos \pi)\vec{i} - (\sin \pi)\vec{j} + \vec{k}}{\sqrt{2}} = \boxed{-\frac{1}{\sqrt{2}}\vec{i} + \frac{1}{\sqrt{2}}\vec{k}.}$$

(b) the curvature $\kappa$ when $t = \pi$.

$$\kappa(t) = \frac{|\vec{T}\,'(t)|}{|\vec{r}\,'(t)|} = \frac{\frac{1}{\sqrt{2}}|(-\sin t)\vec{i} - (\cos t)\vec{j}|}{\sqrt{2}} = \frac{\sqrt{\sin^2 t + \cos^2 t}}{2} = \boxed{\frac{1}{2}.}$$

(c) the length of the curve from $t = 0$ to $t = \pi$.

$$L = \int_a^b |\vec{r}\,'(t)|\, dt = \int_0^\pi \sqrt{2}\, dt = \boxed{\pi\sqrt{2}.}$$

**7. (8 points)** Show that $\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$ does not exist.

Along the $y$-axis (i.e., the line $x = 0$),

$$\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2} = \lim_{y\to 0} \frac{0\cdot y}{0^2+y^2} = 0.$$

However, along the line $y = x$,

$$\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2} = \lim_{x\to 0} \frac{x\cdot x}{x^2+x^2} = \lim_{x\to 0} \frac{x^2}{2x^2} = \frac{1}{2}.$$

Since the two limits are not equal, the limit does not exist.

**8. (10 points)** Find the equation of the tangent plane to the surface

$$z = f(x,y) = x^2 + y^2 + \sin xy \qquad \text{at } P_0 = (0,2,4)$$

The equation of the tangent plane is given by the linearization of $f(x,y)$ near $(x_0, y_0) = (0,2)$:

$$z = L(x,y) = f(0,2) + \frac{\partial f}{\partial x}(0,2)\cdot(x-x_0) + \frac{\partial f}{\partial y}(0,2)\cdot(y-y_0).$$

First observe that

$$f(0,2) = 0^2 + 2^2 + \sin(0\cdot 2) = 4;$$

in fact, this piece of information is given to us as the $z$-coordinate of the point $P_0$. Now

$$\frac{\partial f}{\partial x} = 2x + y\cos xy, \quad \text{so} \quad \frac{\partial f}{\partial x}(0,2) = 2\cdot 0 + 2\cdot\cos(0\cdot 2) = 2;$$

$$\frac{\partial f}{\partial y} = 2y + x\cos xy, \quad \text{so} \quad \frac{\partial f}{\partial y}(0,2) = 2\cdot 2 + 0\cdot\cos(0\cdot 2) = 4.$$

Putting this all together, we obtain

$$z = 4 + 2(x-0) + 4(y-2) = 2x + 4y - 8; \text{ or}$$

$$\boxed{2x + 4y - z = 8.}$$

**9. (8 points)** Consider the surface defined by

$$\frac{x}{y} + \frac{y}{z} + \frac{z}{x} + 1 = 0.$$

Near the point $(1, -1, 1)$, $z$ is defined implicitly as a function of $x$ and $y$.

(a) Find $\dfrac{\partial z}{\partial x}$.

Differentiating both sides with respect to $x$,

$$\frac{1}{y} - \frac{y}{z^2}\frac{\partial z}{\partial x} + \frac{1}{x}\frac{\partial z}{\partial x} - \frac{z}{x^2} = 0.$$

Rearranging and solving for $\frac{\partial z}{\partial x}$,

$$\frac{\partial z}{\partial x}\left(\frac{1}{x} - \frac{y}{z^2}\right) = \frac{z}{x^2} - \frac{1}{y};$$

$$\frac{\partial z}{\partial x} = \boxed{\frac{\frac{z}{x^2} - \frac{1}{y}}{\frac{1}{x} - \frac{y}{z^2}}}.$$

At $(1, -1, 1)$,

$$\frac{\partial z}{\partial x} = \frac{1-(-1)}{1-(-1)} = \frac{2}{2} = \boxed{1.}$$

(b) Find $\dfrac{\partial z}{\partial y}$.

Differentiating both sides with respect to $y$,

$$-\frac{x}{y^2} + \frac{1}{z} - \frac{y}{z^2}\frac{\partial z}{\partial y} + \frac{1}{x}\frac{\partial z}{\partial y} = 0.$$

Rearranging and solving for $\frac{\partial z}{\partial y}$,

$$\frac{\partial z}{\partial y}\left(\frac{1}{x} - \frac{y}{z^2}\right) = \frac{x}{y^2} - \frac{1}{z};$$

$$\frac{\partial z}{\partial x} = \boxed{\frac{\frac{x}{y^2} - \frac{1}{z}}{\frac{1}{x} - \frac{y}{z^2}}}.$$

At $(1, -1, 1)$,

$$\frac{\partial z}{\partial x} = \frac{1-1}{1-(-1)} = \frac{0}{2} = \boxed{0.}$$