Math 164: Multidimensional Calculus

Final Exam

December 17, 2016

NAME (please print legibly): SOLUTIONS
Your University ID Number: ______
Indicate your instructor with a check in the appropriate box:

Kleene	TR 12:30-1:45pm	
Salur	MW 3:25-4:40pm	
Gafni	TR 3:25-4:40pm	
Lee	MWF 09:00-09:50am	

- You are responsible for checking that this exam has all 15 pages.
- No calculators, phones, electronic devices, books, notes are allowed during the exam.
- Show all work and justify all answers. Please sign the pledge below.

Pledge of Honesty

I affirm that I will not give or receive any unauthorized help on this exam, and that all work will be my own.

Signature: ______

Part A		
QUESTION	VALUE	SCORE
1	18	
2	16	
3	16	
4	16	
5	18	
6	16	
TOTAL	100	

Part B		
QUESTION	VALUE	SCORE
7	16	
8	16	
9	16	
10	18	
11	16	
12	18	
TOTAL	100	

Part A

1. (18 points)

Consider the vectors

$$\mathbf{a} = \langle 1, -1, 3\rangle, \quad \mathbf{b} = \langle -2, 1, 1\rangle, \quad \mathbf{c} = \langle 1, 0, 5\rangle .$$

Compute the following.

(a) The angle between **a** and **b**.

$$\vec{a}\cdot\vec{b} = 0 \Rightarrow \theta = 90^\circ$$

(b) The projection of **b** onto **c**.

$$\text{proj}_c(\vec{b}) = \left(\frac{\vec{b}\cdot\vec{c}}{|c|^2}\right)\vec{c} = \frac{3}{26}\langle 1,0,5\rangle$$

(c) The area of the parallelogram spanned by **a** and **c**.

$$\text{area}\;(\;) = |\vec{a}\times\vec{c}| = \begin{vmatrix} i & j & k \\ 1 & -1 & 3 \\ 1 & 0 & 5 \end{vmatrix} = |-i-2j+k| = \sqrt{1+4+1}$$
$$=\sqrt{6}$$

2. (16 points) For each of the following statements, circle TRUE or FALSE. No work is required, and there is no partial credit.

(a) The curve $\mathbf{r}(t) = \langle t^3, -t^3, 2t^3 \rangle$ is a line. TRUE FALSE

$= \langle 1, -1, 2 \rangle t^3$

(b) If $\mathbf{u}(t)$ and $\mathbf{v}(t)$ are differentiable vector functions, then $\frac{d}{dt}[\mathbf{u}(t) \cdot \mathbf{v}(t)] = \mathbf{u}'(t) \cdot \mathbf{v}'(t)$. TRUE FALSE

product rule.

(c) If $|\mathbf{r}(t)| = 1$ for all t then $|\mathbf{r}'(t)| = 0$. TRUE FALSE

ex/ $\vec{r}(t) = \langle \cos(t), \sin(t) \rangle$

(d) The curve $\mathbf{r}(t) = \langle \cos(t), \sin(t), t \rangle$, $0 \leq t \leq 1$, has arclength 4π. TRUE FALSE

$$AL = \int_0^1 \sqrt{\sin^2(t) + \cos^2(t) + 1}\, dt = \sqrt{2}$$

3. (16 points) Find the limit, if it exists, or show that the limit does not exist.

(a)

$$\lim_{(x,y)\to(1,1)} \frac{e^x \ln y}{x^2 + 2y^2} = 0$$

$f(x,y)$ is continuous near $(1,1)$ and

$e^x \ln(y) \to e^1 \ln(1) = e^1 \cdot 0 = 0$

while $x^2 + 2y^2 \to 1 + 2 = 3 \neq 0$

(b)

$$\lim_{(x,y)\to(0,0)} \frac{x \sin y}{x^2 + 2y^2}$$

along the line $y = 0$, $f(x,y) = \frac{0}{x^2+0} \to 0$.

along the line $y = x$, $f(x,y) = \frac{x\sin(x)}{3x^2} = \frac{\sin(x)}{3x} \to \frac{1}{3}$.

Thus, $\lim_{(x,y)\to(0,0)} f(x,y)$ DNE.

4. (16 points) (a) Find the equation of the tangent plane to the surface $z = x^2 + 2y^2$ at the point (2,0,1).

$\nabla z = \langle 2x, 4y \rangle \Rightarrow \nabla z(2,0) = \langle 4, 0 \rangle$

Should be (2,0,4)

<u>tangent plane</u> : $z - z_0 = \nabla z(x_0, y_0) \cdot \langle x - x_0, y - y_0 \rangle$

$z - 4 = \langle 4, 0 \rangle \langle x - 2, y - 0 \rangle$

$z - 4 = 4(x - 2)$

$\boxed{z = 4x - 4}$

(b) What is an approximate value of $f(2.1, -0.1)$ when $f(x, y) = x^2 + 2y^2$?

$L(x, y) = 4x - 8$ at $(2, 0)$

$f(2.1, -0.1) \approx L(2.1, -0.1) = 4(2.1) - 8 = .4$

$\boxed{f(2.1, -0.1) \approx .4}$

5. (18 points) Find the extreme values of the function $f(x, y) = xy$ over the curve $x^2 + y^4 = 3/4$. $g(x,y) = x^2 + y^4$

use Lagrange multipliers:

$$\nabla f = \langle y, x \rangle$$
$$\nabla g = \langle 2x, 4y^3 \rangle$$

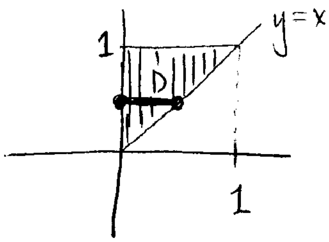
① $y = \lambda 2x$
② $x = \lambda 4y^3$
③ $x^2 + y^4 = \frac{3}{4}$

① and ② $\Rightarrow$ either $(x,y) = (0,0)$,
which is not on the constraint
or $\frac{y}{2x} = \frac{x}{4y^3} \Rightarrow x^2 = 2y^4$
③ $\Rightarrow 3y^4 = \frac{3}{4} \Rightarrow y^4 = \frac{1}{4}$ and $x^2 = \frac{1}{2}$

So the points to consider
are $\left(\frac{1}{2}, \frac{1}{\sqrt{2}}\right), \left(\frac{1}{2}, \frac{-1}{\sqrt{2}}\right), \left(\frac{-1}{2}, \frac{-1}{\sqrt{2}}\right), \left(\frac{-1}{2}, \frac{1}{\sqrt{2}}\right)$

$$\boxed{\begin{array}{l} \max = \frac{1}{2\sqrt{2}} \\ \min = \frac{-1}{2\sqrt{2}} \end{array}}$$

6. (16 points) Evaluate the integral by reversing the order of integration.



$$\int_0^1 \int_x^1 e^{x/y}\,dy\,dx$$

$$= \int_0^1 \int_0^y e^{x/y}\,dx\,dy$$

$$= \int_0^1 \left[y e^{x/y} \right]_0^y dy$$

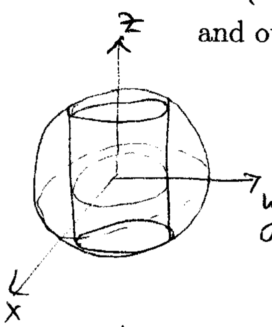
$$= \int_0^1 \left[y e^1 - y e^0 \right] dy$$

$$= (e-1)\,\tfrac{1}{2} y^2 \Big]_0^1$$

$$= \tfrac{1}{2}(e-1)$$

Part B

7. (16 points) Find the volume of the solid that lies inside the sphere $x^2 + y^2 + z^2 = 4$ and outside the cylinder $x^2 + y^2 = 1$.



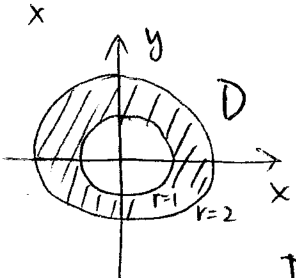
$$V = 2\iint_D \sqrt{4-r^2}\, r\,dr\,d\theta = 2\int_0^{2\pi} \left[-\frac{1}{\cancel{2}}\cdot\frac{\cancel{2}}{3}(4-r^2)^{3/2} \right]_1^2 d\theta$$

$u = 4-r^2$

$-\frac{1}{2}du = r\,dr$

$$= 2\cdot 2\pi \cdot \frac{-1}{3}\left(0 - 3^{3/2}\right)$$

$$= \frac{4\pi}{3}\cdot 3^{3/2}$$



$$D = \{(r,\theta) \mid 0 \le \theta \le 2\pi,\ 1 \le r \le 2\}$$

8. (16 points) Evaluate the line integral $\int_C xyz\,ds$, where C is the line segment from $(1, 2, 3)$ to $(2, 4, 5)$.

$$\vec{r}(t) = (1-t)\langle 1,2,3\rangle + t\langle 2,4,5\rangle \quad \text{for } 0 \le t \le 1$$

or C is given by $\left\{\begin{array}{l} x = 1 - t + 2t = 1+t \\ y = (1-t)2 + 4t = 2+2t \\ z = (1-t)3 + 5t = 3+2t \end{array}\right\}$ so $ds = \sqrt{1^2+2^2+2^2}\,dt = 3\,dt$

$$\int_C xyz\,ds = \int_0^1 (1+t)(2+2t)(3+2t)\,3\,dt$$

$$= 6\int_0^1 (3 + 8t + 7t^2 + 2t^3)\,dt$$

$$= 6\left[3t + 4t^2 + \tfrac{7}{3}t^3 + \tfrac{1}{2}t^4\right]_0^1$$

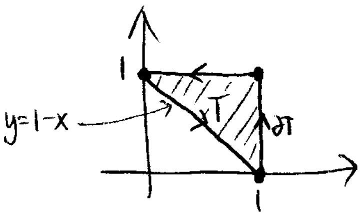
$$= \boxed{59}$$

9. (16 points) Let T be the triangle with vertices $(1,0)$, $(1,1)$ and $\cancel{(1,0)}$, let $\mathbf{F}$ be the vector field given by

Should be $(0,1)$

$$\mathbf{F}(x,y) = \langle xy^2\sin(x^2) + 4yx^2,\ -y\cos(x^2)\rangle.$$
$$= \langle P, Q \rangle$$

Compute $\oint_{\partial T} \mathbf{F}\cdot \mathbf{dr}$.



$T = \{(x,y) \mid 0 \le x \le 1,\ 1-x \le y \le 1\}$

Green's Thm

$$\oint_{\partial T} \vec{F}\cdot d\vec{r} \overset{\downarrow}{=} \iint_T \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

$$= \iint_T \left(\cancel{y\sin(x^2)2x} - \cancel{2xy\sin(x^2)} + 4x^2\right) dA$$

$$= \int_0^1\int_{1-x}^1 4x^2\, dy\, dx$$

$$= \int_0^1 4x^2(1-(1-x))\, dx$$

$$= \int_0^1 4x^3\, dx$$

$$= x^4\Big]_0^1$$

$$= 1$$

10. (18 points) (a) Find a potential function for the vector field

$$\mathbf{F}(x, y, z) = \langle yz + 2xy,\ xz + x^2,\ xy + 4z\rangle = \nabla f \quad \text{for}$$

$f(x,y,z) = xyz + x^2y + 2z^2 + C$ (can take $C = 0$) by eyeballing it.

Otherwise, we $f_x = yz + 2xy \Rightarrow f = xyz + x^2y + g(y,z)$

$f_y = xz + x^2 + g_y = xz + x^2 \Rightarrow g_y = 0 \Rightarrow f = xyz + x^2y + g(z)$

$f_z = xy + 0 + g'(z) \Rightarrow g'(z) = 4z \Rightarrow g(z) = 2z^2 + C$

(b) Evaluate the line integral $\int_C \mathbf{F} \cdot \mathbf{dr}$, where C is the oriented curve parametrized by $r(t) = \langle t, t^2, t^4 - 1\rangle$ for $0 \leq t \leq 1$. $\vec{r}(0) = \langle 0, 0, -1\rangle$, $\vec{r}(1) = \langle 1, 1, 0\rangle$

$\int_C \vec{F} \cdot d\vec{r} = \int_C \nabla f\, dr = f(1,1,0) - f(0,0,-1)$ by F.T. of L.I.s

$= 1 - 2 = \boxed{-1}$

11. (16 points) Evaluate the surface integral $\iint x^2yz\,dS$, where the surface S is the part of the plane $\underbrace{z = 1 + 2x + 3y}_{=g(x,y)}$ that lies above the rectangle $[0,3] \times [0,2]$. $= D$

$$\iint_S f(x,y,z)\,dS = \iint_D f(x,y,g(x,y))\sqrt{\left(\frac{dz}{dx}\right)^2 + \left(\frac{dz}{dy}\right)^2 + 1}\;dA$$

$$= \int_0^3 \int_0^2 x^2y\,(1+2x+3y)\sqrt{2^2+3^2+1}\;dy\,dx$$

$$= \sqrt{14}\int_0^3 \int_0^2 (x^2y + 2x^3y + 3x^2y^2)\,dy\,dx$$

$$= \sqrt{14}\int_0^3 \left(\frac{1}{2}x^2y^2 + x^3y^2 + x^2y^3\right)_0^2 dx$$

$$= \sqrt{14}\int_0^3 (10x^2 + 4x^3)\,dx$$

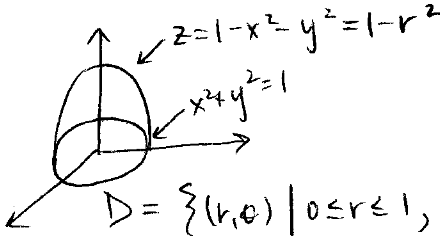
$$= \sqrt{14}\left[\frac{10}{3}x^3 + x^4\right]_0^3$$

$$= \sqrt{14}\,(171)$$

12. (18 points) Compute the flux of the vector field

$$\mathbf{F}(x, y, z) = y\mathbf{i} + x\mathbf{j} + z\mathbf{k}.$$

through the surface S given by the boundary of the solid region E enclosed by the paraboloid $z = 1 - x^2 - y^2$ and the plane $z = 0$. Here S is given the positive (outward) orientation with respect to E.



$D = \{(r,\theta) \mid 0 \le r \le 1,\ 0 \le \theta \le 2\pi\}$

① $F = \langle P, Q, R \rangle$ with P, Q, R having continuous partials on all of $\mathbb{R}^3$ ✓

② $\operatorname{div} \vec{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} = 0 + 0 + 1 = 1$

③ flux of $\vec{F}$ through $S = \iint_S \vec{F} \cdot d\vec{S} \underset{\substack{\uparrow \\ \text{DIV THM}}}{=} \iiint_E \operatorname{div} \vec{F}\, dV = \iiint_E 1\, dV$

$$= \int_0^{2\pi} \int_0^1 (1 - r^2)\, r\, dr\, d\theta = 2\pi \cdot \left[\tfrac{1}{2}r^2 - \tfrac{1}{4}r^4\right]_0^1 = 2\pi \cdot \tfrac{1}{4} = \boxed{\tfrac{\pi}{2}}$$