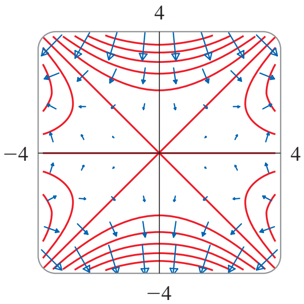


FIGURE 15

EXAMPLE 6 Find the gradient vector field of $f(x, y) = x^2y - y^3$. Plot the gradient vector field together with a contour map of f. How are they related?

SOLUTION The gradient vector field is given by

$$\nabla f(x, y) = \frac{\partial f}{\partial x}\,\mathbf{i} + \frac{\partial f}{\partial y}\,\mathbf{j} = 2xy\,\mathbf{i} + (x^2 - 3y^2)\,\mathbf{j}$$

Figure 15 shows a contour map of f with the gradient vector field. Notice that the gradient vectors are perpendicular to the level curves, as we would expect from Section 14.6. Notice also that the gradient vectors are long where the level curves are close to each other and short where the curves are farther apart. That's because the length of the gradient vector is the value of the directional derivative of f and closely spaced level curves indicate a steep graph. ■

A vector field $\mathbf{F}$ is called a **conservative vector field** if it is the gradient of some scalar function, that is, if there exists a function f such that $\mathbf{F} = \nabla f$. In this situation f is called a **potential function** for $\mathbf{F}$.

Not all vector fields are conservative, but such fields do arise frequently in physics. For example, the gravitational field $\mathbf{F}$ in Example 4 is conservative because if we define

$$f(x, y, z) = \frac{mMG}{\sqrt{x^2 + y^2 + z^2}}$$

then

$$\begin{aligned}\nabla f(x, y, z) &= \frac{\partial f}{\partial x}\,\mathbf{i} + \frac{\partial f}{\partial y}\,\mathbf{j} + \frac{\partial f}{\partial z}\,\mathbf{k}\\ &= \frac{-mMGx}{(x^2 + y^2 + z^2)^{3/2}}\,\mathbf{i} + \frac{-mMGy}{(x^2 + y^2 + z^2)^{3/2}}\,\mathbf{j} + \frac{-mMGz}{(x^2 + y^2 + z^2)^{3/2}}\,\mathbf{k}\\ &= \mathbf{F}(x, y, z)\end{aligned}$$

In Sections 16.3 and 16.5 we will learn how to tell whether or not a given vector field is conservative.

16.1 Exercises

1–12 Sketch the vector field $\mathbf{F}$ by drawing a diagram like Figure 5 or Figure 9.

1. $\mathbf{F}(x, y) = \mathbf{i} + \frac{1}{2}\,\mathbf{j}$

2. $\mathbf{F}(x, y) = 2\,\mathbf{i} - \mathbf{j}$

3. $\mathbf{F}(x, y) = \mathbf{i} + \frac{1}{2}y\,\mathbf{j}$

4. $\mathbf{F}(x, y) = x\,\mathbf{i} + \frac{1}{2}y\,\mathbf{j}$

5. $\mathbf{F}(x, y) = -\frac{1}{2}\,\mathbf{i} + (y - x)\,\mathbf{j}$

6. $\mathbf{F}(x, y) = y\,\mathbf{i} + (x + y)\,\mathbf{j}$

7. $\mathbf{F}(x, y) = \dfrac{y\,\mathbf{i} + x\,\mathbf{j}}{\sqrt{x^2 + y^2}}$

8. $\mathbf{F}(x, y) = \dfrac{y\,\mathbf{i} - x\,\mathbf{j}}{\sqrt{x^2 + y^2}}$

9. $\mathbf{F}(x, y, z) = \mathbf{i}$

10. $\mathbf{F}(x, y, z) = z\,\mathbf{i}$

11. $\mathbf{F}(x, y, z) = -y\,\mathbf{i}$

12. $\mathbf{F}(x, y, z) = \mathbf{i} + \mathbf{k}$

13–18 Match the vector fields $\mathbf{F}$ with the plots labeled I–VI. Give reasons for your choices.

13. $\mathbf{F}(x, y) = \langle x, -y \rangle$

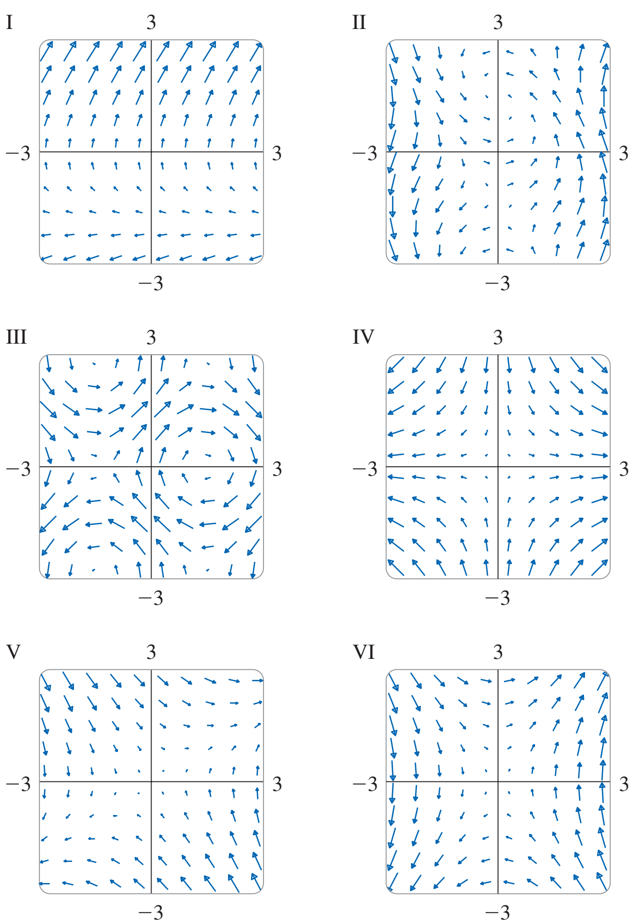
14. $\mathbf{F}(x, y) = \langle y, x - y \rangle$

15. $\mathbf{F}(x, y) = \langle y, y + 2 \rangle$

16. $\mathbf{F}(x, y) = \langle y, 2x \rangle$

17. $\mathbf{F}(x, y) = \langle \sin y, \cos x \rangle$

18. $\mathbf{F}(x, y) = \langle \cos(x + y), x \rangle$



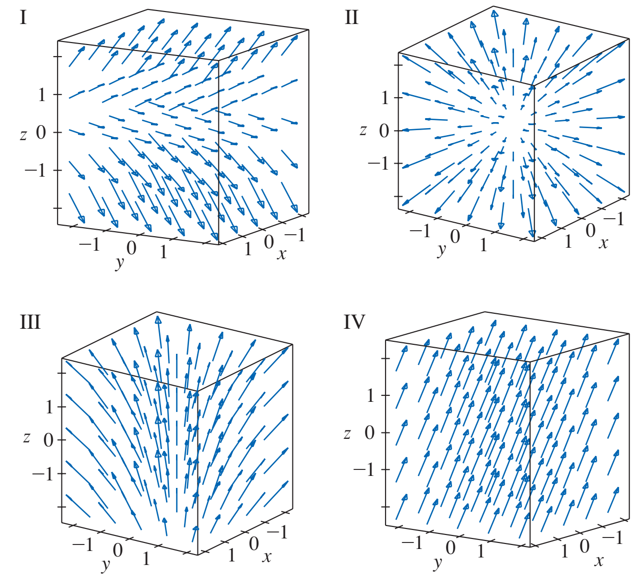
19–22 Match the vector fields $\mathbf{F}$ on $\mathbb{R}^3$ with the plots labeled I–IV. Give reasons for your choices.

19. $\mathbf{F}(x, y, z) = \mathbf{i} + 2\,\mathbf{j} + 3\,\mathbf{k}$

20. $\mathbf{F}(x, y, z) = \mathbf{i} + 2\,\mathbf{j} + z\,\mathbf{k}$

21. $\mathbf{F}(x, y, z) = x\,\mathbf{i} + y\,\mathbf{j} + 3\,\mathbf{k}$

22. $\mathbf{F}(x, y, z) = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$



23. Use graphing software to plot the vector field

$$\mathbf{F}(x, y) = (y^2 - 2xy)\,\mathbf{i} + (3xy - 6x^2)\,\mathbf{j}$$

Explain the appearance by finding the set of points (x, y) such that $\mathbf{F}(x, y) = \mathbf{0}$.

24. Let $\mathbf{F}(\mathbf{x}) = (r^2 - 2r)\mathbf{x}$, where $\mathbf{x} = \langle x, y \rangle$ and $r = |\mathbf{x}|$. Use graphing software to plot this vector field in various domains until you can see what is happening. Describe the appearance of the plot and explain it by finding the points where $\mathbf{F}(\mathbf{x}) = \mathbf{0}$.

25–28 Find the gradient vector field ∇f of f.

25. $f(x, y) = y \sin(xy)$

26. $f(s, t) = \sqrt{2s + 3t}$

27. $f(x, y, z) = \sqrt{x^2 + y^2 + z^2}$

28. $f(x, y, z) = x^2 y e^{y/z}$

29–30 Find the gradient vector field ∇f of f and sketch it.

29. $f(x, y) = \frac{1}{2}(x - y)^2$

30. $f(x, y) = \frac{1}{2}(x^2 - y^2)$

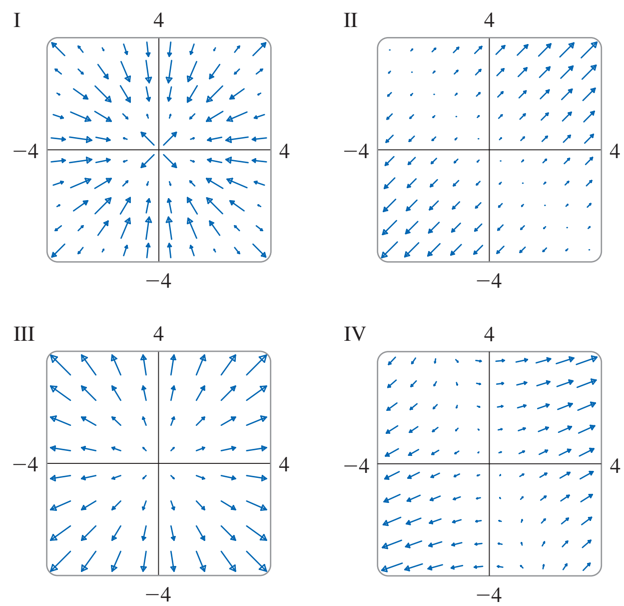
31–34 Match the functions f with the plots of their gradient vector fields labeled I–IV. Give reasons for your choices.

31. $f(x, y) = x^2 + y^2$

32. $f(x, y) = x(x + y)$

33. $f(x, y) = (x + y)^2$

34. $f(x, y) = \sin\sqrt{x^2 + y^2}$



35–36 Plot the gradient vector field of f together with a contour map of f. Explain how they are related to each other.

35. $f(x, y) = \ln(1 + x^2 + 2y^2)$

36. $f(x, y) = \cos x - 2 \sin y$

37. A particle moves in a velocity field $\mathbf{V}(x, y) = \langle x^2, x + y^2 \rangle$. If it is at position $(2, 1)$ at time $t = 3$, estimate its location at time $t = 3.01$.

38. At time $t = 1$, a particle is located at position $(1, 3)$. If it moves in a velocity field

$$\mathbf{F}(x, y) = \langle xy - 2, y^2 - 10 \rangle$$

find its approximate location at time $t = 1.05$.

39–40 Flow Lines The *flow lines* (or *streamlines*) of a vector field are the paths followed by a particle whose velocity field is the given vector field. Thus the vectors in a vector field are tangent to the flow lines.

39. (a) Use a sketch of the vector field $\mathbf{F}(x, y) = x\,\mathbf{i} - y\,\mathbf{j}$ to draw some flow lines. From your sketches, can you guess the equations of the flow lines?

(b) If parametric equations of a flow line are $x = x(t)$, $y = y(t)$, explain why these functions satisfy the differential equations $dx/dt = x$ and $dy/dt = -y$. Then solve the differential equations to find an equation of the flow line that passes through the point $(1, 1)$.

40. (a) Sketch the vector field $\mathbf{F}(x, y) = \mathbf{i} + x\,\mathbf{j}$ and then sketch some flow lines. What shape do these flow lines appear to have?

(b) If parametric equations of the flow lines are $x = x(t)$, $y = y(t)$, what differential equations do these functions satisfy? Deduce that $dy/dx = x$.

(c) If a particle starts at the origin in the velocity field given by $\mathbf{F}$, find an equation of the path it follows.

16.2 Line Integrals

In this section we define an integral that is similar to a single integral except that instead of integrating over an interval $[a, b]$, we integrate over a curve C. Such integrals are called *line integrals,* although "curve integrals" would be better terminology. They were invented in the early 19th century to solve problems involving fluid flow, forces, electricity, and magnetism.

Line Integrals in the Plane

We start with a plane curve C given by the parametric equations

$$\boxed{1} \qquad x = x(t) \qquad y = y(t) \qquad a \leqslant t \leqslant b$$

or, equivalently, by the vector equation $\mathbf{r}(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j}$, and we assume that C is a smooth curve. [This means that $\mathbf{r}'$ is continuous and $\mathbf{r}'(t) \neq \mathbf{0}$. See Section 13.3.] If we divide the parameter interval $[a, b]$ into n subintervals $[t_{i-1}, t_i]$ of equal width and we let $x_i = x(t_i)$ and $y_i = y(t_i)$, then the corresponding points $P_i(x_i, y_i)$ divide C into n subarcs with lengths $\Delta s_1, \Delta s_2, \ldots, \Delta s_n$. (See Figure 1.) We choose any point $P_i^*(x_i^*, y_i^*)$ in the ith subarc. (This corresponds to a point t_i^* in $[t_{i-1}, t_i]$.) Now if f is any function of two

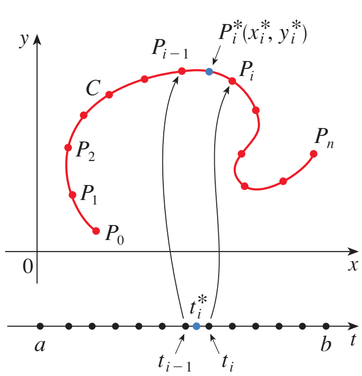


FIGURE 1

But this last integral is precisely the line integral in (10). Therefore we have

$$\boxed{\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C P\,dx + Q\,dy + R\,dz \qquad \text{where } \mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j} + R\,\mathbf{k}}$$

For example, the integral $\int_C y\,dx + z\,dy + x\,dz$ in Example 6 could be expressed as $\int_C \mathbf{F} \cdot d\mathbf{r}$, where

$$\mathbf{F}(x, y, z) = y\,\mathbf{i} + z\,\mathbf{j} + x\,\mathbf{k}$$

A similar result holds for vector fields $\mathbf{F}$ on $\mathbb{R}^2$:

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C P\,dx + Q\,dy \tag{14}$$

where $\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j}$.

16.2 Exercises

1–8 Evaluate the line integral, where C is the given plane curve.

1. $\int_C y\,ds$, C: $x = t^2$, $y = 2t$, $0 \leqslant t \leqslant 3$

2. $\int_C (x/y)\,ds$, C: $x = t^3$, $y = t^4$, $1 \leqslant t \leqslant 2$

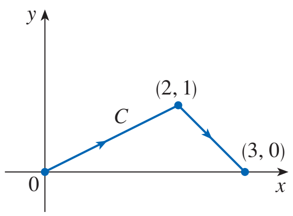
3. $\int_C xy^4\,ds$, C is the right half of the circle $x^2 + y^2 = 16$

4. $\int_C xe^y\,ds$, C is the line segment from $(2, 0)$ to $(5, 4)$

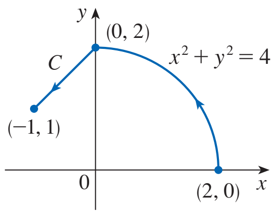
5. $\int_C (x^2y + \sin x)\,dy$,
C is the arc of the parabola $y = x^2$ from $(0, 0)$ to (π, π^2)

6. $\int_C e^x\,dx$,
C is the arc of the curve $x = y^3$ from $(-1, -1)$ to $(1, 1)$

7. $\int_C (x + 2y)\,dx + x^2\,dy$



8. $\int_C x^2\,dx + y^2\,dy$



9–18 Evaluate the line integral, where C is the given space curve.

9. $\int_C x^2y\,ds$,
C: $x = \cos t$, $y = \sin t$, $z = t$, $0 \leqslant t \leqslant \pi/2$

10. $\int_C y^2z\,ds$,
C is the line segment from $(3, 1, 2)$ to $(1, 2, 5)$

11. $\int_C xe^{yz}\,ds$,
C is the line segment from $(0, 0, 0)$ to $(1, 2, 3)$

12. $\int_C (x^2 + y^2 + z^2)\,ds$,
C: $x = t$, $y = \cos 2t$, $z = \sin 2t$, $0 \leqslant t \leqslant 2\pi$

13. $\int_C xye^{yz}\,dy$, C: $x = t$, $y = t^2$, $z = t^3$, $0 \leqslant t \leqslant 1$

14. $\int_C ye^z\,dz + x \ln x\,dy - y\,dx$,
C: $x = e^t$, $y = 2t$, $z = \ln t$, $1 \leqslant t \leqslant 2$

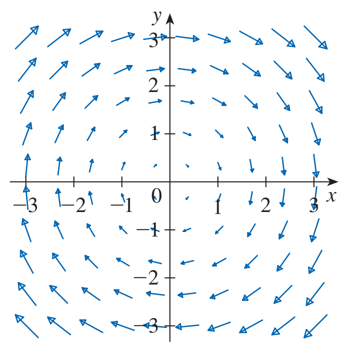
15. $\int_C z\,dx + xy\,dy + y^2\,dz$,
C: $x = \sin t$, $y = \cos t$, $z = \tan t$, $-\pi/4 \leqslant t \leqslant \pi/4$

16. $\int_C y\,dx + z\,dy + x\,dz$,
C: $x = \sqrt{t}$, $y = t$, $z = t^2$, $1 \leqslant t \leqslant 4$

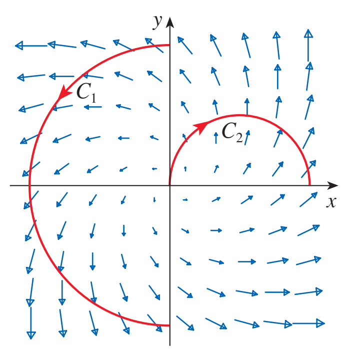
17. $\int_C z^2\,dx + x^2\,dy + y^2\,dz$,
C is the line segment from $(1, 0, 0)$ to $(4, 1, 2)$

18. $\int_C (y + z)\,dx + (x + z)\,dy + (x + y)\,dz$,
C consists of line segments from $(0, 0, 0)$ to $(1, 0, 1)$ and from $(1, 0, 1)$ to $(0, 1, 2)$

19. Let $\mathbf{F}$ be the vector field shown in the figure.
(a) If C_1 is the vertical line segment from $(-3, -3)$ to $(-3, 3)$, determine whether $\int_{C_1} \mathbf{F} \cdot d\mathbf{r}$ is positive, negative, or zero.
(b) If C_2 is the counterclockwise-oriented circle with radius 3 and center the origin, determine whether $\int_{C_2} \mathbf{F} \cdot d\mathbf{r}$ is positive, negative, or zero.



20. The figure shows a vector field $\mathbf{F}$ and two curves C_1 and C_2. Are the line integrals of $\mathbf{F}$ over C_1 and C_2 positive, negative, or zero? Explain.



21–24 Evaluate the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$, where C is given by the vector function $\mathbf{r}(t)$.

21. $\mathbf{F}(x, y) = xy^2\,\mathbf{i} - x^2\,\mathbf{j}$,
$\mathbf{r}(t) = t^3\,\mathbf{i} + t^2\,\mathbf{j}, \quad 0 \leqslant t \leqslant 1$

22. $\mathbf{F}(x, y, z) = (x + y^2)\,\mathbf{i} + xz\,\mathbf{j} + (y + z)\,\mathbf{k}$,
$\mathbf{r}(t) = t^2\,\mathbf{i} + t^3\,\mathbf{j} - 2t\,\mathbf{k}, \quad 0 \leqslant t \leqslant 2$

23. $\mathbf{F}(x, y, z) = \sin x\,\mathbf{i} + \cos y\,\mathbf{j} + xz\,\mathbf{k}$,
$\mathbf{r}(t) = t^3\,\mathbf{i} - t^2\,\mathbf{j} + t\,\mathbf{k}, \quad 0 \leqslant t \leqslant 1$

24. $\mathbf{F}(x, y, z) = xz\,\mathbf{i} + z^3\,\mathbf{j} + y\,\mathbf{k}$,
$\mathbf{r}(t) = e^t\,\mathbf{i} + e^{2t}\,\mathbf{j} + e^{-t}\,\mathbf{k}, \quad -1 \leqslant t \leqslant 1$

T **25–28** Use a calculator or computer to evaluate the line integral correct to four decimal places.

25. $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F}(x, y) = \sqrt{x + y}\,\mathbf{i} + (y/x)\,\mathbf{j}$ and $\mathbf{r}(t) = \sin^2 t\,\mathbf{i} + \sin t \cos t\,\mathbf{j}, \ \pi/6 \leqslant t \leqslant \pi/3$

26. $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F}(x, y, z) = yze^x\,\mathbf{i} + zxe^y\,\mathbf{j} + xye^z\,\mathbf{k}$ and $\mathbf{r}(t) = \sin t\,\mathbf{i} + \cos t\,\mathbf{j} + \tan t\,\mathbf{k}, \ 0 \leqslant t \leqslant \pi/4$

27. $\int_C xy \arctan z\, ds$, where C has parametric equations $x = t^2, y = t^3, z = \sqrt{t}, \ 1 \leqslant t \leqslant 2$

28. $\int_C z \ln(x + y)\, ds$, where C has parametric equations $x = 1 + 3t, y = 2 + t^2, z = t^4, \ -1 \leqslant t \leqslant 1$

29–30 Use a graph of the vector field $\mathbf{F}$ and the curve C to guess whether the line integral of $\mathbf{F}$ over C is positive, negative, or zero. Then evaluate the line integral.

29. $\mathbf{F}(x, y) = (x - y)\,\mathbf{i} + xy\,\mathbf{j}$,
C is the arc of the circle $x^2 + y^2 = 4$ traversed counterclockwise from $(2, 0)$ to $(0, -2)$

30. $\mathbf{F}(x, y) = \dfrac{x}{\sqrt{x^2 + y^2}}\,\mathbf{i} + \dfrac{y}{\sqrt{x^2 + y^2}}\,\mathbf{j}$,
C is the parabola $y = 1 + x^2$ from $(-1, 2)$ to $(1, 2)$

31. (a) Evaluate the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F}(x, y) = e^{x-1}\,\mathbf{i} + xy\,\mathbf{j}$ and C is given by $\mathbf{r}(t) = t^2\,\mathbf{i} + t^3\,\mathbf{j}, \ 0 \leqslant t \leqslant 1$.
(b) Illustrate part (a) by graphing C and the vectors from the vector field corresponding to $t = 0, 1/\sqrt{2}$, and 1 (as in Figure 14).

32. (a) Evaluate the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F}(x, y, z) = x\,\mathbf{i} - z\,\mathbf{j} + y\,\mathbf{k}$ and C is given by $\mathbf{r}(t) = 2t\,\mathbf{i} + 3t\,\mathbf{j} - t^2\,\mathbf{k}, \ -1 \leqslant t \leqslant 1$.
(b) Illustrate part (a) by graphing C and the vectors from the vector field corresponding to $t = \pm 1$ and $\pm\frac{1}{2}$ (as in Figure 14).

T **33.** Use a computer algebra system to find the exact value of $\int_C x^3 y^2 z\, ds$, where C is the curve with parametric equations $x = e^{-t} \cos 4t, \ y = e^{-t} \sin 4t, \ z = e^{-t}, \ 0 \leqslant t \leqslant 2\pi$.

34. (a) Find the work done by the force field $\mathbf{F}(x, y) = x^2\,\mathbf{i} + xy\,\mathbf{j}$ on a particle that moves once around the circle $x^2 + y^2 = 4$ oriented in the counterclockwise direction.
(b) Graph the force field and circle on the same screen. Use the graph to explain your answer to part (a).

35. A thin wire is bent into the shape of a semicircle $x^2 + y^2 = 4, x \geqslant 0$. If the linear density is a constant k, find the mass and center of mass of the wire.

36. A thin wire has the shape of the first-quadrant portion of the circle with center the origin and radius a. If the density function is $\rho(x, y) = kxy$, find the mass and center of mass of the wire.

37. (a) Write the formulas similar to Equations 4 for the center of mass $(\bar{x}, \bar{y}, \bar{z})$ of a thin wire in the shape of a space curve C if the wire has density function $\rho(x, y, z)$.

(b) Find the center of mass of a wire in the shape of the helix $x = 2\sin t$, $y = 2\cos t$, $z = 3t$, $0 \leqslant t \leqslant 2\pi$, if the density is a constant k.

38. Find the mass and center of mass of a wire in the shape of the helix $x = t$, $y = \cos t$, $z = \sin t$, $0 \leqslant t \leqslant 2\pi$, if the density at any point is equal to the square of the distance from the origin.

39. If a wire with linear density $\rho(x, y)$ lies along a plane curve C, its **moments of inertia** about the x- and y-axes are defined as

$$I_x = \int_C y^2 \rho(x, y)\, ds \qquad I_y = \int_C x^2 \rho(x, y)\, ds$$

Find the moments of inertia for the wire in Example 3.

40. If a wire with linear density $\rho(x, y, z)$ lies along a space curve C, its **moments of inertia** about the x-, y-, and z-axes are defined as

$$I_x = \int_C (y^2 + z^2)\rho(x, y, z)\, ds$$

$$I_y = \int_C (x^2 + z^2)\rho(x, y, z)\, ds$$

$$I_z = \int_C (x^2 + y^2)\rho(x, y, z)\, ds$$

Find the moments of inertia for the wire in Exercise 37(b).

41. Find the work done by the force field

$$\mathbf{F}(x, y) = x\,\mathbf{i} + (y + 2)\,\mathbf{j}$$

in moving an object along an arch of the cycloid

$$\mathbf{r}(t) = (t - \sin t)\,\mathbf{i} + (1 - \cos t)\,\mathbf{j} \qquad 0 \leqslant t \leqslant 2\pi$$

42. Find the work done by the force field $\mathbf{F}(x, y) = x^2\,\mathbf{i} + ye^x\,\mathbf{j}$ on a particle that moves along the parabola $x = y^2 + 1$ from $(1, 0)$ to $(2, 1)$.

43. Find the work done by the force field

$$\mathbf{F}(x, y, z) = \langle x - y^2, y - z^2, z - x^2 \rangle$$

on a particle that moves along the line segment from $(0, 0, 1)$ to $(2, 1, 0)$.

44. The force exerted by an electric charge at the origin on a charged particle at a point (x, y, z) with position vector $\mathbf{r} = \langle x, y, z \rangle$ is $\mathbf{F}(\mathbf{r}) = K\mathbf{r}/|\mathbf{r}|^3$ where K is a constant. (See Example 16.1.5.) Find the work done as the particle moves along a straight line from $(2, 0, 0)$ to $(2, 1, 5)$.

45. The position of an object with mass m at time t is $\mathbf{r}(t) = at^2\,\mathbf{i} + bt^3\,\mathbf{j}$, $0 \leqslant t \leqslant 1$.
(a) What is the force acting on the object at time t?
(b) What is the work done by the force during the time interval $0 \leqslant t \leqslant 1$?

46. An object with mass m moves with position function $\mathbf{r}(t) = a\sin t\,\mathbf{i} + b\cos t\,\mathbf{j} + ct\,\mathbf{k}$, $0 \leqslant t \leqslant \pi/2$. Find the work done on the object during this time period.

47. A 160-lb man carries a 25-lb can of paint up a helical staircase that encircles a silo with a radius of 20 ft. If the silo is 90 ft high and the man makes exactly three complete revolutions climbing to the top, how much work is done by the man against gravity?

48. Suppose there is a hole in the can of paint in Exercise 47 and 9 lb of paint leaks steadily out of the can during the man's ascent. How much work is done?

49. (a) Show that a constant force field does zero work on a particle that moves once uniformly around the circle $x^2 + y^2 = 1$.
(b) Is this also true for a force field $\mathbf{F}(\mathbf{x}) = k\mathbf{x}$, where k is a constant and $\mathbf{x} = \langle x, y \rangle$?

50. The base of a circular fence with radius 10 m is given by $x = 10\cos t$, $y = 10\sin t$. The height of the fence at position (x, y) is given by the function $h(x, y) = 4 + 0.01(x^2 - y^2)$, so the height varies from 3 m to 5 m. Suppose that 1 L of paint covers 100 m^2. Sketch the fence and determine how much paint you will need if you paint both sides of the fence.

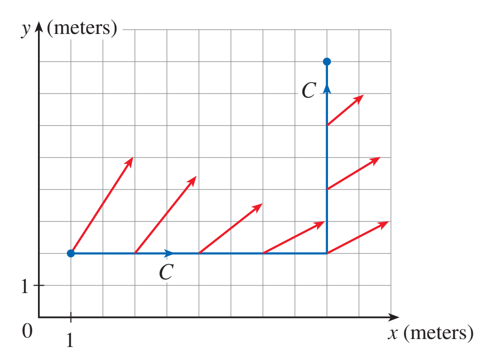
51. If C is a smooth curve given by a vector function $\mathbf{r}(t)$, $a \leqslant t \leqslant b$, and $\mathbf{v}$ is a constant vector, show that

$$\int_C \mathbf{v} \cdot d\mathbf{r} = \mathbf{v} \cdot [\mathbf{r}(b) - \mathbf{r}(a)]$$

52. If C is a smooth curve given by a vector function $\mathbf{r}(t)$, $a \leqslant t \leqslant b$, show that

$$\int_C \mathbf{r} \cdot d\mathbf{r} = \tfrac{1}{2}\left[|\mathbf{r}(b)|^2 - |\mathbf{r}(a)|^2\right]$$

53. An object moves along the curve C shown in the figure from $(1, 2)$ to $(9, 8)$. The lengths of the vectors in the force field $\mathbf{F}$ are measured in newtons by the scales on the axes. Estimate the work done by $\mathbf{F}$ on the object.

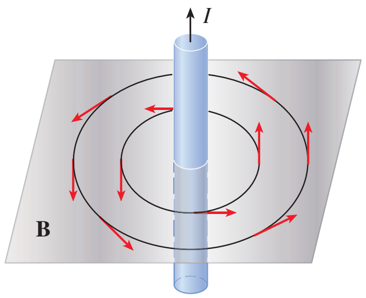


54. Experiments show that a steady current I in a long wire produces a magnetic field $\mathbf{B}$ that is tangent to any circle that lies in the plane perpendicular to the wire and whose center is the axis of the wire (as in the figure). *Ampère's Law* relates the electric current to its magnetic effects and states that

$$\int_C \mathbf{B} \cdot d\mathbf{r} = \mu_0 I$$

where I is the net current that passes through any surface bounded by a closed curve C, and μ_0 is a constant called the permeability of free space. By taking C to be a circle with radius r, show that the magnitude $B = |\mathbf{B}|$ of the magnetic field at a distance r from the center of the wire is

$$B = \frac{\mu_0 I}{2\pi r}$$



16.3 The Fundamental Theorem for Line Integrals

Recall from Section 5.3 that Part 2 of the Fundamental Theorem of Calculus can be written as

$$\boxed{1} \qquad \int_a^b F'(x)\,dx = F(b) - F(a)$$

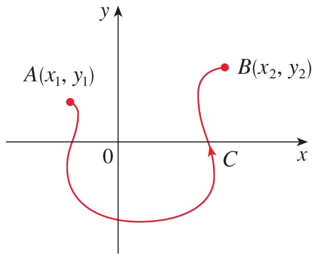
where F' is continuous on $[a, b]$. Equation 1 says that to evaluate the definite integral of F' on $[a, b]$, we need only know the values of F at a and b, the endpoints of the interval. In this section we formulate a similar result for line integrals.

The Fundamental Theorem for Line Integrals

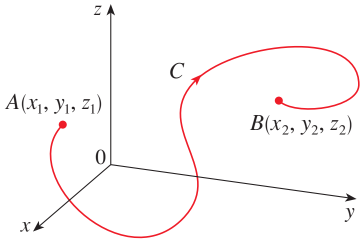
If we think of the gradient vector ∇f of a function f of two or three variables as a sort of derivative of f, then the following theorem can be regarded as a version of the Fundamental Theorem for line integrals.

2 Theorem Let C be a smooth curve given by the vector function $\mathbf{r}(t)$, $a \leqslant t \leqslant b$. Let f be a differentiable function of two or three variables whose gradient vector ∇f is continuous on C. Then

$$\int_C \nabla f \cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a))$$



(a)



(b)

FIGURE 1

NOTE 1 Theorem 2 says that we can evaluate the line integral of a conservative vector field (the gradient vector field of the potential function f) simply by knowing the value of f at the endpoints of C. In fact, Theorem 2 says that the line integral of ∇f is the net change in f. If f is a function of two variables and C is a plane curve with initial point $A(x_1, y_1)$ and terminal point $B(x_2, y_2)$, as in Figure 1(a), then Theorem 2 becomes

$$\int_C \nabla f \cdot d\mathbf{r} = f(x_2, y_2) - f(x_1, y_1)$$

If f is a function of three variables and C is a space curve joining the point $A(x_1, y_1, z_1)$ to the point $B(x_2, y_2, z_2)$, as in Figure 1(b), then we have

$$\int_C \nabla f \cdot d\mathbf{r} = f(x_2, y_2, z_2) - f(x_1, y_1, z_1)$$

So the work done by the force on the object is

$$W = \int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt = \int_a^b m\mathbf{r}''(t) \cdot \mathbf{r}'(t)\, dt$$

$$= \frac{m}{2} \int_a^b \frac{d}{dt} [\mathbf{r}'(t) \cdot \mathbf{r}'(t)]\, dt \qquad \text{(Theorem 13.2.3, Formula 4)}$$

$$= \frac{m}{2} \int_a^b \frac{d}{dt} |\mathbf{r}'(t)|^2\, dt = \frac{m}{2} \Big[|\mathbf{r}'(t)|^2 \Big]_a^b \qquad \text{(Fundamental Theorem of Calculus)}$$

$$= \frac{m}{2} \left(|\mathbf{r}'(b)|^2 - |\mathbf{r}'(a)|^2 \right)$$

Therefore

$$\boxed{15} \qquad W = \tfrac{1}{2} m |\mathbf{v}(b)|^2 - \tfrac{1}{2} m |\mathbf{v}(a)|^2$$

where $\mathbf{v} = \mathbf{r}'$ is the velocity.

The quantity $\frac{1}{2} m |\mathbf{v}(t)|^2$, that is, half the mass times the square of the speed, is called the **kinetic energy** of the object. Therefore we can rewrite Equation 15 as

$$\boxed{16} \qquad W = K(B) - K(A)$$

which says that the work done by the force field along C is equal to the change in kinetic energy at the endpoints of C.

Now let's further assume that $\mathbf{F}$ is a conservative force field; that is, we can write $\mathbf{F} = \nabla f$. In physics, the **potential energy** of an object at the point (x, y, z) is defined as $P(x, y, z) = -f(x, y, z)$, so we have $\mathbf{F} = -\nabla P$. Then by Theorem 2 we have

$$W = \int_C \mathbf{F} \cdot d\mathbf{r} = -\int_C \nabla P \cdot d\mathbf{r} = -[P(\mathbf{r}(b)) - P(\mathbf{r}(a))] = P(A) - P(B)$$

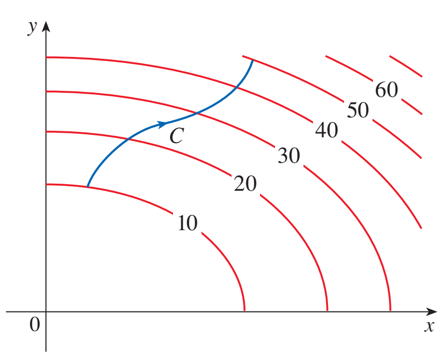
Comparing this equation with Equation 16, we see that

$$P(A) + K(A) = P(B) + K(B)$$

which says that if an object moves from one point A to another point B under the influence of a conservative force field, then the sum of its potential energy and its kinetic energy remains constant. This is called the **Law of Conservation of Energy** and it is the reason the vector field is called *conservative*.

16.3 Exercises

1. The figure shows a curve C and a contour map of a function f whose gradient is continuous. Find $\int_C \nabla f \cdot d\mathbf{r}$.



2. A table of values of a function f with continuous gradient is given. Find $\int_C \nabla f \cdot d\mathbf{r}$, where C has parametric equations

$$x = t^2 + 1 \qquad y = t^3 + t \qquad 0 \leqslant t \leqslant 1$$

$x \backslash y$	0	1	2
0	1	6	4
1	3	5	7
2	8	2	9

3–10 Determine whether or not $\mathbf{F}$ is a conservative vector field. If it is, find a function f such that $\mathbf{F} = \nabla f$.

3. $\mathbf{F}(x, y) = (xy + y^2)\,\mathbf{i} + (x^2 + 2xy)\,\mathbf{j}$

4. $\mathbf{F}(x, y) = (y^2 - 2x)\,\mathbf{i} + 2xy\,\mathbf{j}$

5. $\mathbf{F}(x, y) = y^2 e^{xy}\,\mathbf{i} + (1 + xy)e^{xy}\,\mathbf{j}$

6. $\mathbf{F}(x, y) = ye^x\,\mathbf{i} + (e^x + e^y)\,\mathbf{j}$

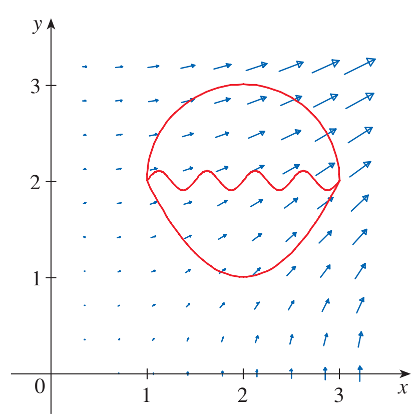
7. $\mathbf{F}(x, y) = (ye^x + \sin y)\,\mathbf{i} + (e^x + x\cos y)\,\mathbf{j}$

8. $\mathbf{F}(x, y) = (2xy + y^{-2})\,\mathbf{i} + (x^2 - 2xy^{-3})\,\mathbf{j}, \quad y > 0$

9. $\mathbf{F}(x, y) = (y^2\cos x + \cos y)\,\mathbf{i} + (2y\sin x - x\sin y)\,\mathbf{j}$

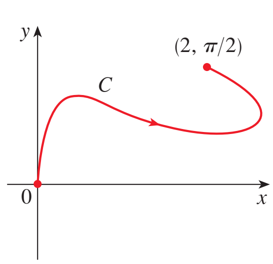
10. $\mathbf{F}(x, y) = (\ln y + y/x)\,\mathbf{i} + (\ln x + x/y)\,\mathbf{j}$

11. The figure shows the vector field $\mathbf{F}(x, y) = \langle 2xy, x^2 \rangle$ and three curves that start at (1, 2) and end at (3, 2).
(a) Explain why $\int_C \mathbf{F} \cdot d\mathbf{r}$ has the same value for all three curves.
(b) What is this common value?

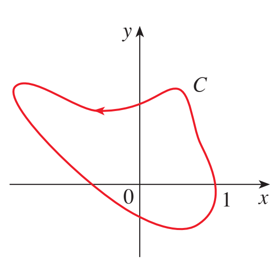


12. Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ for the vector field $\mathbf{F}(x, y) = 2xy\,\mathbf{i} + (x^2 + \sin y)\,\mathbf{j}$ and the curve C shown.

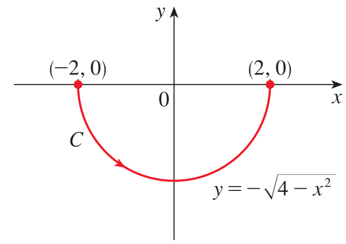
(a)



(b)



13. Let $\mathbf{F}(x, y) = (3x^2 + y^2)\,\mathbf{i} + 2xy\,\mathbf{j}$ and let C be the curve shown.



(a) Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ directly.
(b) Show that $\mathbf{F}$ is conservative and find a function f such that $\mathbf{F} = \nabla f$.
(c) Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ using Theorem 2.
(d) Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ by first replacing C by a simpler curve that has the same initial and terminal points.

14–15 A vector field $\mathbf{F}$ and a curve C are given.
(a) Show that $\mathbf{F}$ is conservative and find a potential function f.
(b) Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ using Theorem 2.
(c) Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ by first replacing C with a line segment that has the same initial and terminal points.

14. $\mathbf{F}(x, y) = \langle \sin y + e^x, x\cos y \rangle$,
C: $x = t,\; y = t(3 - t),\; 0 \leqslant t \leqslant 3$

15. $\mathbf{F}(x, y) = \langle ye^{xy}, xe^{xy} \rangle$,
C: $x = \sin \dfrac{\pi}{2} t,\; y = e^{t-1}(1 - \cos \pi t),\; 0 \leqslant t \leqslant 1$

16. Evaluate $\int_C \nabla f \cdot d\mathbf{r}$, where $f(x, y, z) = xy^2 z + x^2$ and C is the curve $x = t^2$, $y = e^{t^2 - 1}$, $z = t^2 + t$, $-1 \leqslant t \leqslant 1$.

17–24 (a) Find a function f such that $\mathbf{F} = \nabla f$ and (b) use part (a) to evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ along the given curve C.

17. $\mathbf{F}(x, y) = \langle 2x, 4y \rangle$,
C is the arc of the parabola $x = y^2$ from $(4, -2)$ to $(1, 1)$

18. $\mathbf{F}(x, y) = (3 + 2xy^2)\,\mathbf{i} + 2x^2y\,\mathbf{j}$,
C is the arc of the hyperbola $y = 1/x$ from (1, 1) to $\left(4, \frac{1}{4}\right)$

19. $\mathbf{F}(x, y) = x^2y^3\,\mathbf{i} + x^3y^2\,\mathbf{j}$,
C: $\mathbf{r}(t) = \langle t^3 - 2t, t^3 + 2t \rangle, \quad 0 \leqslant t \leqslant 1$

20. $\mathbf{F}(x, y) = (1 + xy)e^{xy}\,\mathbf{i} + x^2e^{xy}\,\mathbf{j}$,
C: $\mathbf{r}(t) = \cos t\,\mathbf{i} + 2\sin t\,\mathbf{j}, \quad 0 \leqslant t \leqslant \pi/2$

21. $\mathbf{F}(x, y, z) = 2xy\,\mathbf{i} + (x^2 + 2yz)\,\mathbf{j} + y^2\,\mathbf{k}$,
C is the line segment from $(2, -3, 1)$ to $(-5, 1, 2)$

22. $\mathbf{F}(x, y, z) = (y^2z + 2xz^2)\,\mathbf{i} + 2xyz\,\mathbf{j} + (xy^2 + 2x^2z)\,\mathbf{k}$,
C: $x = \sqrt{t},\; y = t + 1,\; z = t^2, \quad 0 \leqslant t \leqslant 1$

23. $\mathbf{F}(x, y, z) = yze^{xz}\,\mathbf{i} + e^{xz}\,\mathbf{j} + xye^{xz}\,\mathbf{k}$,
C: $\mathbf{r}(t) = (t^2 + 1)\,\mathbf{i} + (t^2 - 1)\,\mathbf{j} + (t^2 - 2t)\,\mathbf{k}$,
$0 \leqslant t \leqslant 2$

24. $\mathbf{F}(x, y, z) = \sin y\,\mathbf{i} + (x\cos y + \cos z)\,\mathbf{j} - y\sin z\,\mathbf{k}$,
C: $\mathbf{r}(t) = \sin t\,\mathbf{i} + t\,\mathbf{j} + 2t\,\mathbf{k}$, $\quad 0 \leqslant t \leqslant \pi/2$

25–26 Show that the line integral is independent of path and evaluate the integral.

25. $\int_C 2xe^{-y}\,dx + (2y - x^2e^{-y})\,dy$,
C is any path from $(1, 0)$ to $(2, 1)$

26. $\int_C \sin y\,dx + (x\cos y - \sin y)\,dy$,
C is any path from $(2, 0)$ to $(1, \pi)$

27. Suppose you're asked to determine the curve that requires the least work for a force field $\mathbf{F}$ to move a particle from one point to another point. You decide to check first whether $\mathbf{F}$ is conservative, and indeed it turns out that it is. How would you reply to the request?

28. Suppose an experiment determines that the amount of work required for a force field $\mathbf{F}$ to move a particle from the point $(1, 2)$ to the point $(5, -3)$ along a curve C_1 is 1.2 J and the work done by $\mathbf{F}$ in moving the particle along another curve C_2 between the same two points is 1.4 J. What can you say about $\mathbf{F}$? Why?

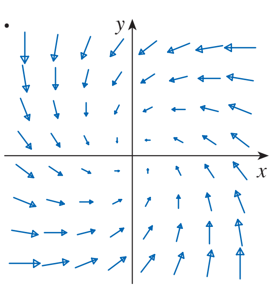
29–30 Find the work done by the force field $\mathbf{F}$ in moving an object from P to Q.

29. $\mathbf{F}(x, y) = x^3\,\mathbf{i} + y^3\,\mathbf{j}$; $\quad P(1, 0), Q(2, 2)$

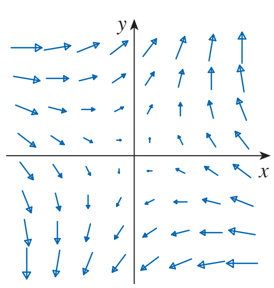
30. $\mathbf{F}(x, y) = (2x + y)\,\mathbf{i} + x\,\mathbf{j}$; $\quad P(1, 1), Q(4, 3)$

31–32 Is the vector field shown in the figure conservative? Explain.

31.



32.



33. If $\mathbf{F}(x, y) = \sin y\,\mathbf{i} + (1 + x\cos y)\,\mathbf{j}$, use a plot to guess whether $\mathbf{F}$ is conservative. Then determine whether your guess is correct.

34. Let $\mathbf{F} = \nabla f$, where $f(x, y) = \sin(x - 2y)$. Find curves C_1 and C_2 that are not closed and satisfy the equation.

(a) $\displaystyle\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = 0$ $\qquad$ (b) $\displaystyle\int_{C_2} \mathbf{F} \cdot d\mathbf{r} = 1$

35. Show that if the vector field $\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j} + R\,\mathbf{k}$ is conservative and P, Q, R have continuous first-order partial derivatives, then

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \qquad \frac{\partial P}{\partial z} = \frac{\partial R}{\partial x} \qquad \frac{\partial Q}{\partial z} = \frac{\partial R}{\partial y}$$

36. Use Exercise 35 to show that the line integral $\int_C y\,dx + x\,dy + xyz\,dz$ is not independent of path.

37–40 Determine whether or not the given set is (a) open, (b) connected, and (c) simply-connected.

37. $\{(x, y) \mid 0 < y < 3\}$

38. $\{(x, y) \mid 1 < |x| < 2\}$

39. $\{(x, y) \mid 1 \leqslant x^2 + y^2 \leqslant 4,\ y \geqslant 0\}$

40. $\{(x, y) \mid (x, y) \neq (2, 3)\}$

41. Let $\mathbf{F}(x, y) = \dfrac{-y\,\mathbf{i} + x\,\mathbf{j}}{x^2 + y^2}$.

(a) Show that $\partial P/\partial y = \partial Q/\partial x$.
(b) Show that $\int_C \mathbf{F} \cdot d\mathbf{r}$ is not independent of path. [*Hint:* Compute $\int_{C_1} \mathbf{F} \cdot d\mathbf{r}$ and $\int_{C_2} \mathbf{F} \cdot d\mathbf{r}$, where C_1 and C_2 are the upper and lower halves of the circle $x^2 + y^2 = 1$ from $(1, 0)$ to $(-1, 0)$.] Does this contradict Theorem 6?

42. Inverse Square Fields Suppose that $\mathbf{F}$ is an *inverse square force field*, that is,

$$\mathbf{F}(\mathbf{r}) = \frac{c\mathbf{r}}{|\mathbf{r}|^3}$$

for some constant c, where $\mathbf{r} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$.

(a) Find the work done by $\mathbf{F}$ in moving an object from a point P_1 along a path to a point P_2 in terms of the distances d_1 and d_2 from these points to the origin.
(b) An example of an inverse square field is the gravitational field $\mathbf{F} = -(mMG)\mathbf{r}/|\mathbf{r}|^3$ discussed in Example 16.1.4. Use part (a) to find the work done by the gravitational field when the earth moves from aphelion (at a maximum distance of 1.52×10^8 km from the sun) to perihelion (at a minimum distance of 1.47×10^8 km). (Use the values $m = 5.97 \times 10^{24}$ kg, $M = 1.99 \times 10^{30}$ kg, and $G = 6.67 \times 10^{-11}\ \mathrm{N{\cdot}m^2/kg^2}$.)
(c) Another example of an inverse square field is the electric force field $\mathbf{F} = \varepsilon qQ\mathbf{r}/|\mathbf{r}|^3$ discussed in Example 16.1.5. Suppose that an electron with a charge of -1.6×10^{-19} C is located at the origin. A positive unit charge is positioned a distance 10^{-12} m from the electron and moves to a position half that distance from the electron. Use part (a) to find the work done by the electric force field. (Use the value $\varepsilon = 8.985 \times 10^9$.)

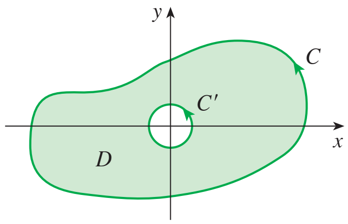


FIGURE 11

with center the origin and radius a, where a is chosen to be small enough that C' lies inside C. (See Figure 11.) Let D be the region bounded by C and C'. Then its positively oriented boundary is $C \cup (-C')$ and so the general version of Green's Theorem gives

$$\int_C P\,dx + Q\,dy + \int_{-C'} P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

$$= \iint_D \left[\frac{y^2 - x^2}{(x^2+y^2)^2} - \frac{y^2 - x^2}{(x^2+y^2)^2}\right] dA = 0$$

Therefore $$\int_C P\,dx + Q\,dy = \int_{C'} P\,dx + Q\,dy$$

that is, $$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_{C'} \mathbf{F} \cdot d\mathbf{r}$$

We now easily compute this last integral using the parametrization given by $\mathbf{r}(t) = a \cos t\,\mathbf{i} + a \sin t\,\mathbf{j}$, $0 \leqslant t \leqslant 2\pi$. Thus

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_{C'} \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\,dt$$

$$= \int_0^{2\pi} \frac{(-a \sin t)(-a \sin t) + (a \cos t)(a \cos t)}{a^2 \cos^2 t + a^2 \sin^2 t}\,dt = \int_0^{2\pi} dt = 2\pi$$ ■

We end this section by using Green's Theorem to discuss a result that was stated in the preceding section.

SKETCH OF PROOF OF THEOREM 16.3.6 We're assuming that $\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j}$ is a vector field on an open simply-connected region D, that P and Q have continuous first-order partial derivatives, and that

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \qquad \text{throughout } D$$

If C is any simple closed path in D and R is the region that C encloses, then Green's Theorem gives

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \oint_C P\,dx + Q\,dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = \iint_R 0\,dA = 0$$

A curve that is not simple crosses itself at one or more points and can be broken up into a number of simple curves. We have shown that the line integrals of $\mathbf{F}$ around these simple curves are all 0 and, adding these integrals, we see that $\int_C \mathbf{F} \cdot d\mathbf{r} = 0$ for any closed curve C. Therefore $\int_C \mathbf{F} \cdot d\mathbf{r}$ is independent of path in D by Theorem 16.3.3. It follows that $\mathbf{F}$ is a conservative vector field. ■

16.4 Exercises

1–4 Evaluate the line integral by two methods: (a) directly and (b) using Green's Theorem.

1. $\oint_C y^2\,dx + x^2 y\,dy$,
C is the rectangle with vertices (0, 0), (5, 0), (5, 4), and (0, 4)

2. $\oint_C y\,dx - x\,dy$,
C is the circle with center the origin and radius 4

3. $\oint_C xy\,dx + x^2 y^3\,dy$,
C is the triangle with vertices (0, 0), (1, 0), and (1, 2)

4. $\oint_C x^2y^2\,dx + xy\,dy$, C consists of the arc of the parabola $y = x^2$ from $(0, 0)$ to $(1, 1)$ and the line segments from $(1, 1)$ to $(0, 1)$ and from $(0, 1)$ to $(0, 0)$

5–12 Use Green's Theorem to evaluate the line integral along the given positively oriented curve.

5. $\int_C ye^x\,dx + 2e^x\,dy$,
C is the rectangle with vertices $(0, 0)$, $(3, 0)$, $(3, 4)$, and $(0, 4)$

6. $\int_C \ln(xy)\,dx + (y/x)\,dy$,
C is the rectangle with vertices $(1, 1)$, $(1, 4)$, $(2, 4)$, and $(2, 1)$

7. $\int_C x^2y^2\,dx + y\tan^{-1}y\,dy$,
C is the triangle with vertices $(0, 0)$, $(1, 0)$, and $(1, 3)$

8. $\int_C (x^2 + y^2)\,dx + (x^2 - y^2)\,dy$,
C is the triangle with vertices $(0, 0)$, $(2, 1)$, and $(0, 1)$

9. $\int_C \left(y + e^{\sqrt{x}}\right)dx + (2x + \cos y^2)\,dy$,
C is the boundary of the region enclosed by the parabolas $y = x^2$ and $x = y^2$

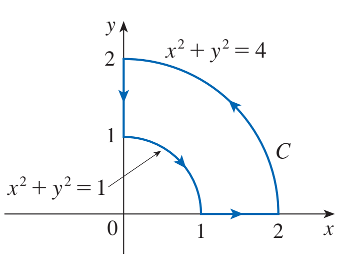
10. $\int_C y^4\,dx + 2xy^3\,dy$, C is the ellipse $x^2 + 2y^2 = 2$

11. $\int_C y^3\,dx - x^3\,dy$, C is the circle $x^2 + y^2 = 4$

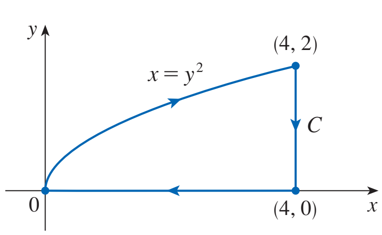
12. $\int_C (1 - y^3)\,dx + (x^3 + e^{y^2})\,dy$, C is the boundary of the region between the circles $x^2 + y^2 = 4$ and $x^2 + y^2 = 9$

13–18 Use Green's Theorem to evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$. (Check the orientation of the curve before applying the theorem.)

13. $\int_C (3 + e^{x^2})\,dx + (\tan^{-1}y + 3x^2)\,dy$



14. $\int_C (x^{2/3} + y^2)\,dx + (y^{4/3} - x^2)\,dy$



15. $\mathbf{F}(x, y) = \langle y\cos x - xy\sin x,\, xy + x\cos x\rangle$,
C is the triangle from $(0, 0)$ to $(0, 4)$ to $(2, 0)$ to $(0, 0)$

16. $\mathbf{F}(x, y) = \langle e^{-x} + y^2,\, e^{-y} + x^2\rangle$,
C consists of the arc of the curve $y = \cos x$ from $(-\pi/2, 0)$ to $(\pi/2, 0)$ and the line segment from $(\pi/2, 0)$ to $(-\pi/2, 0)$

17. $\mathbf{F}(x, y) = \langle y - \cos y,\, x\sin y\rangle$,
C is the circle $(x - 3)^2 + (y + 4)^2 = 4$ oriented clockwise

18. $\mathbf{F}(x, y) = \left\langle \sqrt{x^2 + 1},\, \tan^{-1}x\right\rangle$, C is the triangle from $(0, 0)$ to $(1, 1)$ to $(0, 1)$ to $(0, 0)$

19–20 Verify Green's Theorem by using a computer algebra system to evaluate both the line integral and the double integral.

19. $P(x, y) = x^3y^4$, $Q(x, y) = x^5y^4$,
C consists of the line segment from $(-\pi/2, 0)$ to $(\pi/2, 0)$ followed by the arc of the curve $y = \cos x$ from $(\pi/2, 0)$ to $(-\pi/2, 0)$

20. $P(x, y) = 2x - x^3y^5$, $Q(x, y) = x^3y^8$,
C is the ellipse $4x^2 + y^2 = 4$

21. Use Green's Theorem to find the work done by the force $\mathbf{F}(x, y) = x(x + y)\,\mathbf{i} + xy^2\,\mathbf{j}$ in moving a particle from the origin along the x-axis to $(1, 0)$, then along the line segment to $(0, 1)$, and then back to the origin along the y-axis.

22. A particle starts at the origin, moves along the x-axis to $(5, 0)$, then along the quarter-circle $x^2 + y^2 = 25$, $x \geqslant 0$, $y \geqslant 0$ to the point $(0, 5)$, and then down the y-axis back to the origin. Use Green's Theorem to find the work done on this particle by the force field $\mathbf{F}(x, y) = \left\langle \sin x,\, \sin y + xy^2 + \tfrac{1}{3}x^3\right\rangle$.

23. Use one of the formulas in (5) to find the area under one arch of the cycloid $x = t - \sin t$, $y = 1 - \cos t$.

24. If a circle C with radius 1 rolls along the outside of the circle $x^2 + y^2 = 16$, a fixed point P on C traces out a curve called an *epicycloid*, with parametric equations $x = 5\cos t - \cos 5t$, $y = 5\sin t - \sin 5t$. Graph the epicycloid and use (5) to find the area it encloses.

25. (a) If C is the line segment connecting the point (x_1, y_1) to the point (x_2, y_2), show that

$$\int_C x\,dy - y\,dx = x_1y_2 - x_2y_1$$

(b) If the vertices of a polygon, in counterclockwise order, are (x_1, y_1), (x_2, y_2), . . . , (x_n, y_n), show that the area of the polygon is

$$A = \tfrac{1}{2}[(x_1y_2 - x_2y_1) + (x_2y_3 - x_3y_2) + \cdots + (x_{n-1}y_n - x_ny_{n-1}) + (x_ny_1 - x_1y_n)]$$

(c) Find the area of the pentagon with vertices $(0, 0)$, $(2, 1)$, $(1, 3)$, $(0, 2)$, and $(-1, 1)$.

26. Let D be a region bounded by a simple closed path C in the xy-plane. Use Green's Theorem to prove that the coordinates of the centroid $(\bar{x}, \bar{y})$ of D are

$$\bar{x} = \frac{1}{2A} \oint_C x^2\, dy \qquad \bar{y} = -\frac{1}{2A} \oint_C y^2\, dx$$

where A is the area of D.

27. Use Exercise 26 to find the centroid of a quarter-circular region of radius a.

28. Use Exercise 26 to find the centroid of the triangle with vertices $(0, 0)$, $(a, 0)$, and (a, b), where $a > 0$ and $b > 0$.

29. A plane lamina with constant density $\rho(x, y) = \rho$ occupies a region in the xy-plane bounded by a simple closed path C. Show that its moments of inertia about the axes are

$$I_x = -\frac{\rho}{3} \oint_C y^3\, dx \qquad I_y = \frac{\rho}{3} \oint_C x^3\, dy$$

(See Section 15.4.)

30. Use Exercise 29 to find the moment of inertia of a circular disk of radius a with constant density ρ about a diameter. (Compare with Example 15.4.4.)

31. Use the method of Example 5 to calculate $\int_C \mathbf{F} \cdot d\mathbf{r}$, where

$$\mathbf{F}(x, y) = \frac{2xy\,\mathbf{i} + (y^2 - x^2)\,\mathbf{j}}{(x^2 + y^2)^2}$$

and C is any positively oriented simple closed curve that encloses the origin.

32. Calculate $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F}(x, y) = \langle x^2 + y, 3x - y^2 \rangle$ and C is the positively oriented boundary curve of a region D that has area 6.

33. If $\mathbf{F}$ is the vector field of Example 5, show that $\int_C \mathbf{F} \cdot d\mathbf{r} = 0$ for every simple closed path that does not pass through or enclose the origin.

34. Complete the proof of the special case of Green's Theorem by proving Equation 3.

35. Use Green's Theorem to prove the change of variables formula for a double integral (Formula 15.9.9) for the case where $f(x, y) = 1$:

$$\iint_R dx\, dy = \iint_S \left| \frac{\partial(x, y)}{\partial(u, v)} \right| du\, dv$$

Here R is the region in the xy-plane that corresponds to the region S in the uv-plane under the transformation given by $x = g(u, v)$, $y = h(u, v)$.

[*Hint:* Note that the left side is $A(R)$ and apply the first part of Equation 5. Convert the line integral over ∂R to a line integral over ∂S and apply Green's Theorem in the uv-plane.]

16.5 Curl and Divergence

In this section we define two operations that can be performed on vector fields and that play a basic role in the applications of vector calculus to fluid flow and electricity and magnetism. Each operation resembles differentiation, but one produces a vector field whereas the other produces a scalar field.

Curl

If $\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j} + R\,\mathbf{k}$ is a vector field on $\mathbb{R}^3$ and the partial derivatives of P, Q, and R all exist, then the curl of $\mathbf{F}$ is the vector field on $\mathbb{R}^3$ defined by

$$\boxed{1} \qquad \operatorname{curl} \mathbf{F} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right) \mathbf{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \right) \mathbf{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \mathbf{k}$$

As an aid to our memory, let's rewrite Equation 1 using operator notation. We introduce the vector differential operator ∇ ("del") as

$$\nabla = \mathbf{i} \frac{\partial}{\partial x} + \mathbf{j} \frac{\partial}{\partial y} + \mathbf{k} \frac{\partial}{\partial z}$$

If C is given by the vector equation

$$\mathbf{r}(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j} \qquad a \leqslant t \leqslant b$$

then the unit tangent vector (see Section 13.2) is

$$\mathbf{T}(t) = \frac{x'(t)}{|\mathbf{r}'(t)|}\,\mathbf{i} + \frac{y'(t)}{|\mathbf{r}'(t)|}\,\mathbf{j}$$

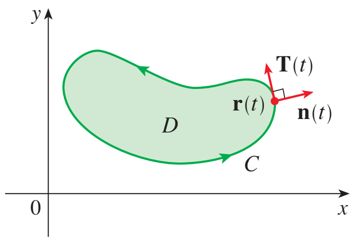


FIGURE 4

You can verify that the outward unit normal vector to C is given by

$$\mathbf{n}(t) = \frac{y'(t)}{|\mathbf{r}'(t)|}\,\mathbf{i} - \frac{x'(t)}{|\mathbf{r}'(t)|}\,\mathbf{j}$$

(See Figure 4.) Then, from Equation 16.2.3, we have

$$\begin{aligned}
\oint_C \mathbf{F}\cdot\mathbf{n}\,ds &= \int_a^b (\mathbf{F}\cdot\mathbf{n})(t)\,|\mathbf{r}'(t)|\,dt \\
&= \int_a^b \left[\frac{P(x(t), y(t))\,y'(t)}{|\mathbf{r}'(t)|} - \frac{Q(x(t), y(t))\,x'(t)}{|\mathbf{r}'(t)|}\right]|\mathbf{r}'(t)|\,dt \\
&= \int_a^b P(x(t), y(t))\,y'(t)\,dt - Q(x(t), y(t))\,x'(t)\,dt \\
&= \int_C P\,dy - Q\,dx = \iint_D \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\right) dA
\end{aligned}$$

by Green's Theorem. But the integrand in this double integral is just the divergence of $\mathbf{F}$. So we have a second vector form of Green's Theorem.

13
$$\oint_C \mathbf{F}\cdot\mathbf{n}\,ds = \iint_D \operatorname{div}\mathbf{F}(x, y)\,dA$$

This version says that the line integral of the normal component of $\mathbf{F}$ along C is equal to the double integral of the divergence of $\mathbf{F}$ over the region D enclosed by C.

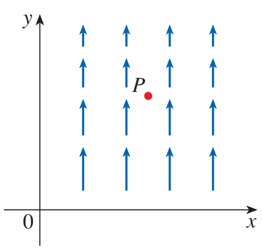
16.5 Exercises

1–8 Find (a) the curl and (b) the divergence of the vector field.

1. $\mathbf{F}(x, y, z) = xy^2z^2\,\mathbf{i} + x^2yz^2\,\mathbf{j} + x^2y^2z\,\mathbf{k}$
2. $\mathbf{F}(x, y, z) = x^3yz^2\,\mathbf{j} + y^4z^3\,\mathbf{k}$
3. $\mathbf{F}(x, y, z) = xye^z\,\mathbf{i} + yze^x\,\mathbf{k}$
4. $\mathbf{F}(x, y, z) = \sin yz\,\mathbf{i} + \sin zx\,\mathbf{j} + \sin xy\,\mathbf{k}$
5. $\mathbf{F}(x, y, z) = \dfrac{\sqrt{x}}{1+z}\,\mathbf{i} + \dfrac{\sqrt{y}}{1+x}\,\mathbf{j} + \dfrac{\sqrt{z}}{1+y}\,\mathbf{k}$
6. $\mathbf{F}(x, y, z) = \ln(2y + 3z)\,\mathbf{i} + \ln(x + 3z)\,\mathbf{j} + \ln(x + 2y)\,\mathbf{k}$
7. $\mathbf{F}(x, y, z) = \langle e^x \sin y,\, e^y \sin z,\, e^z \sin x\rangle$
8. $\mathbf{F}(x, y, z) = \langle \arctan(xy), \arctan(yz), \arctan(zx)\rangle$

9–12 The vector field $\mathbf{F}$ is shown in the xy-plane and looks the same in all other horizontal planes. (In other words, $\mathbf{F}$ is independent of z and its z-component is 0.)
(a) Is div $\mathbf{F}$ positive, negative, or zero at P? Explain.
(b) Determine whether curl $\mathbf{F} = \mathbf{0}$. If not, in which direction does curl $\mathbf{F}$ point at P?

9.



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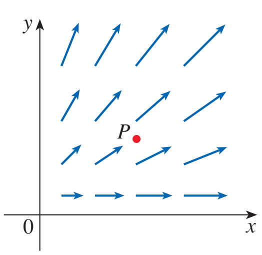

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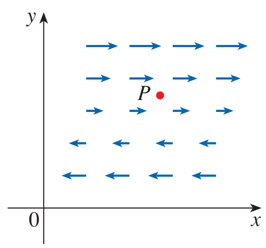

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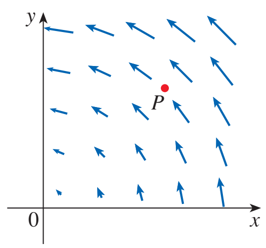


13. (a) Verify Formula 3 for $f(x, y, z) = \sin xyz$.
(b) Verify Formula 11 for $\mathbf{F}(x, y, z) = xyz^2\,\mathbf{i} + x^2yz^3\,\mathbf{j} + y^2\,\mathbf{k}$.

14. Let f be a scalar field and $\mathbf{F}$ a vector field. State whether each expression is meaningful. If not, explain why. If so, state whether it is a scalar field or a vector field.

(a) curl f (b) grad f
(c) div $\mathbf{F}$ (d) curl(grad f)
(e) grad $\mathbf{F}$ (f) grad(div $\mathbf{F}$)
(g) div(grad f) (h) grad(div f)
(i) curl(curl $\mathbf{F}$) (j) div(div $\mathbf{F}$)
(k) (grad f) $\times$ (div $\mathbf{F}$) (l) div(curl(grad f))

15–20 Determine whether or not the vector field is conservative. If it is conservative, find a function f such that $\mathbf{F} = \nabla f$.

15. $\mathbf{F}(x, y, z) = \langle 2xy^3z^2, 3x^2y^2z^2, 2x^2y^3z \rangle$

16. $\mathbf{F}(x, y, z) = \langle yz, xz + y, xy - x \rangle$

17. $\mathbf{F}(x, y, z) = \langle \ln y, (x/y) + \ln z, y/z \rangle$

18. $\mathbf{F}(x, y, z) = yz \sin xy\,\mathbf{i} + xz \sin xy\,\mathbf{j} - \cos xy\,\mathbf{k}$

19. $\mathbf{F}(x, y, z) = yz^2e^{xz}\,\mathbf{i} + ze^{xz}\,\mathbf{j} + xyze^{xz}\,\mathbf{k}$

20. $\mathbf{F}(x, y, z) = e^z \cos x\,\mathbf{i} + e^y \cos z\,\mathbf{j} + (e^z \sin x - e^y \sin z)\,\mathbf{k}$

21. Is there a vector field $\mathbf{G}$ on $\mathbb{R}^3$ such that curl $\mathbf{G} = \langle x \sin y, \cos y, z - xy \rangle$? Explain.

22. Is there a vector field $\mathbf{G}$ on $\mathbb{R}^3$ such that curl $\mathbf{G} = \langle x, y, z \rangle$? Explain.

23. Show that any vector field of the form

$$\mathbf{F}(x, y, z) = f(x)\,\mathbf{i} + g(y)\,\mathbf{j} + h(z)\,\mathbf{k}$$

where f, g, h are differentiable functions, is irrotational.

24. Show that any vector field of the form

$$\mathbf{F}(x, y, z) = f(y, z)\,\mathbf{i} + g(x, z)\,\mathbf{j} + h(x, y)\,\mathbf{k}$$

is incompressible.

25–31 Prove the identity, assuming that the appropriate partial derivatives exist and are continuous. If f is a scalar field and $\mathbf{F}$, $\mathbf{G}$ are vector fields, then $f\mathbf{F}$, $\mathbf{F} \cdot \mathbf{G}$, and $\mathbf{F} \times \mathbf{G}$ are defined by

$$(f\mathbf{F})(x, y, z) = f(x, y, z)\,\mathbf{F}(x, y, z)$$
$$(\mathbf{F} \cdot \mathbf{G})(x, y, z) = \mathbf{F}(x, y, z) \cdot \mathbf{G}(x, y, z)$$
$$(\mathbf{F} \times \mathbf{G})(x, y, z) = \mathbf{F}(x, y, z) \times \mathbf{G}(x, y, z)$$

25. $\operatorname{div}(\mathbf{F} + \mathbf{G}) = \operatorname{div} \mathbf{F} + \operatorname{div} \mathbf{G}$

26. $\operatorname{curl}(\mathbf{F} + \mathbf{G}) = \operatorname{curl} \mathbf{F} + \operatorname{curl} \mathbf{G}$

27. $\operatorname{div}(f\mathbf{F}) = f \operatorname{div} \mathbf{F} + \mathbf{F} \cdot \nabla f$

28. $\operatorname{curl}(f\mathbf{F}) = f \operatorname{curl} \mathbf{F} + (\nabla f) \times \mathbf{F}$

29. $\operatorname{div}(\mathbf{F} \times \mathbf{G}) = \mathbf{G} \cdot \operatorname{curl} \mathbf{F} - \mathbf{F} \cdot \operatorname{curl} \mathbf{G}$

30. $\operatorname{div}(\nabla f \times \nabla g) = 0$

31. $\operatorname{curl}(\operatorname{curl} \mathbf{F}) = \operatorname{grad}(\operatorname{div} \mathbf{F}) - \nabla^2 \mathbf{F}$

32–34 Let $\mathbf{r} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$ and $r = |\mathbf{r}|$.

32. Verify each identity.
(a) $\nabla \cdot \mathbf{r} = 3$ (b) $\nabla \cdot (r\mathbf{r}) = 4r$
(c) $\nabla^2 r^3 = 12r$

33. Verify each identity.
(a) $\nabla r = \mathbf{r}/r$ (b) $\nabla \times \mathbf{r} = \mathbf{0}$
(c) $\nabla(1/r) = -\mathbf{r}/r^3$ (d) $\nabla \ln r = \mathbf{r}/r^2$

34. If $\mathbf{F} = \mathbf{r}/r^p$, find div $\mathbf{F}$. Is there a value of p for which div $\mathbf{F} = 0$?

35. Use Green's Theorem in the form of Equation 13 to prove **Green's first identity**:

$$\iint_D f\nabla^2 g\, dA = \oint_C f(\nabla g) \cdot \mathbf{n}\, ds - \iint_D \nabla f \cdot \nabla g\, dA$$

where D and C satisfy the hypotheses of Green's Theorem and the appropriate partial derivatives of f and g exist and are continuous. (The quantity $\nabla g \cdot \mathbf{n} = D_{\mathbf{n}}\, g$ occurs in the line integral; it is the directional derivative in the direction of the normal vector $\mathbf{n}$ and is called the **normal derivative** of g.)

36. Use Green's first identity (Exercise 35) to prove **Green's second identity**:

$$\iint_D (f\nabla^2 g - g\nabla^2 f)\, dA = \oint_C (f\nabla g - g\nabla f) \cdot \mathbf{n}\, ds$$

where D and C satisfy the hypotheses of Green's Theorem and the appropriate partial derivatives of f and g exist and are continuous.

37. Recall from Section 14.3 that a function g is called *harmonic* on D if it satisfies Laplace's equation, that is, $\nabla^2 g = 0$ on D. Use Green's first identity (with the same hypotheses as in Exercise 35) to show that if g is harmonic on D, then $\oint_C D_{\mathbf{n}}\, g\, ds = 0$. Here $D_{\mathbf{n}}\, g$ is the normal derivative of g defined in Exercise 35.

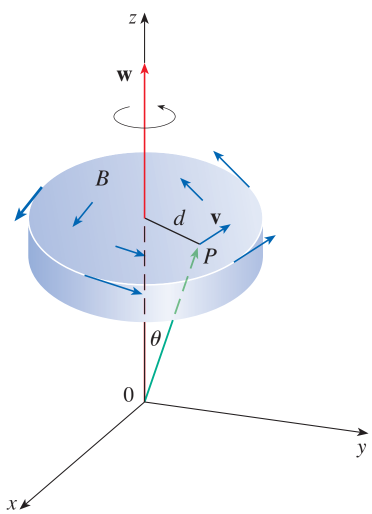
38. Use Green's first identity to show that if f is harmonic on D, and if $f(x, y) = 0$ on the boundary curve C, then $\iint_D |\nabla f|^2\, dA = 0$. (Assume the same hypotheses as in Exercise 35.)

39. This exercise demonstrates a connection between the curl vector and rotations. Let B be a rigid body rotating about the z-axis. The rotation can be described by the vector $\mathbf{w} = \omega \mathbf{k}$, where ω is the angular speed of B, that is, the tangential speed of any point P in B divided by the distance d from the axis of rotation. Let $\mathbf{r} = \langle x, y, z \rangle$ be the position vector of P.

(a) By considering the angle θ in the figure, show that the velocity field of B is given by $\mathbf{v} = \mathbf{w} \times \mathbf{r}$.

(b) Show that $\mathbf{v} = -\omega y\, \mathbf{i} + \omega x\, \mathbf{j}$.

(c) Show that $\operatorname{curl} \mathbf{v} = 2\mathbf{w}$.



40. Maxwell's equations relating the electric field $\mathbf{E}$ and magnetic field $\mathbf{H}$ as they vary with time in a region containing no charge and no current can be stated as follows:

$$\operatorname{div} \mathbf{E} = 0 \qquad \operatorname{div} \mathbf{H} = 0$$

$$\operatorname{curl} \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{H}}{\partial t} \qquad \operatorname{curl} \mathbf{H} = \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t}$$

where c is the speed of light. Use these equations to prove the following:

(a) $\nabla \times (\nabla \times \mathbf{E}) = -\dfrac{1}{c^2}\dfrac{\partial^2 \mathbf{E}}{\partial t^2}$

(b) $\nabla \times (\nabla \times \mathbf{H}) = -\dfrac{1}{c^2}\dfrac{\partial^2 \mathbf{H}}{\partial t^2}$

(c) $\nabla^2 \mathbf{E} = \dfrac{1}{c^2}\dfrac{\partial^2 \mathbf{E}}{\partial t^2}$ [*Hint:* Use Exercise 31.]

(d) $\nabla^2 \mathbf{H} = \dfrac{1}{c^2}\dfrac{\partial^2 \mathbf{H}}{\partial t^2}$

41. We have seen that all vector fields of the form $\mathbf{F} = \nabla g$ satisfy the equation $\operatorname{curl} \mathbf{F} = \mathbf{0}$ and that all vector fields of the form $\mathbf{F} = \operatorname{curl} \mathbf{G}$ satisfy the equation $\operatorname{div} \mathbf{F} = 0$ (assuming continuity of the appropriate partial derivatives). This suggests the question: are there any equations that all functions of the form $f = \operatorname{div} \mathbf{G}$ must satisfy? Show that the answer to this question is "no" by proving that *every* continuous function f on $\mathbb{R}^3$ is the divergence of some vector field.
[*Hint:* Let $\mathbf{G}(x, y, z) = \langle g(x, y, z), 0, 0 \rangle$, where $g(x, y, z) = \int_0^x f(t, y, z)\, dt$.]

16.6 Parametric Surfaces and Their Areas

So far we have considered special types of surfaces: cylinders, quadric surfaces, graphs of functions of two variables, and level surfaces of functions of three variables. Here we use vector functions to describe more general surfaces, called *parametric surfaces*, and compute their areas. Then we take the general surface area formula and see how it applies to special surfaces.

Parametric Surfaces

In much the same way that we describe a space curve by a vector function $\mathbf{r}(t)$ of a single parameter t, we can describe a surface by a vector function $\mathbf{r}(u, v)$ of two parameters u and v. We suppose that

$$\boxed{1} \qquad \mathbf{r}(u, v) = x(u, v)\, \mathbf{i} + y(u, v)\, \mathbf{j} + z(u, v)\, \mathbf{k}$$

is a vector-valued function defined on a region D in the uv-plane. So x, y, and z, the component functions of $\mathbf{r}$, are functions of the two variables u and v with domain D. The set of all points (x, y, z) in $\mathbb{R}^3$ such that

$$\boxed{2} \qquad x = x(u, v) \qquad y = y(u, v) \qquad z = z(u, v)$$

and (u, v) varies throughout D, is called a **parametric surface** S and Equations 2 are called **parametric equations** of S. Each choice of u and v gives a point on S; by making because $f(x) \geqslant 0$. Therefore the area of S is

$$\begin{aligned} A &= \iint_D |\mathbf{r}_x \times \mathbf{r}_\theta|\, dA \\ &= \int_0^{2\pi} \int_a^b f(x)\sqrt{1 + [f'(x)]^2}\, dx\, d\theta \\ &= 2\pi \int_a^b f(x)\sqrt{1 + [f'(x)]^2}\, dx \end{aligned}$$

This is precisely the formula that was used to define the area of a surface of revolution in single-variable calculus (8.2.4).

16.6 Exercises

1–2 Determine whether the points P and Q lie on the given surface.

1. $\mathbf{r}(u, v) = \langle u + v, u - 2v, 3 + u - v \rangle$
$P(4, -5, 1)$, $Q(0, 4, 6)$

2. $\mathbf{r}(u, v) = \langle 1 + u - v, u + v^2, u^2 - v^2 \rangle$
$P(1, 2, 1)$, $Q(2, 3, 3)$

3–6 Identify the surface with the given vector equation.

3. $\mathbf{r}(u, v) = (u + v)\,\mathbf{i} + (3 - v)\,\mathbf{j} + (1 + 4u + 5v)\,\mathbf{k}$

4. $\mathbf{r}(u, v) = u^2\,\mathbf{i} + u\cos v\,\mathbf{j} + u\sin v\,\mathbf{k}$

5. $\mathbf{r}(s, t) = \langle s\cos t, s\sin t, s \rangle$

6. $\mathbf{r}(s, t) = \langle 3\cos t, s, \sin t \rangle$, $\quad -1 \leqslant s \leqslant 1$

7–12 Use a computer to graph the parametric surface. Indicate on the graph which grid curves have u constant and which have v constant.

7. $\mathbf{r}(u, v) = \langle u^2, v^2, u + v \rangle$,
$-1 \leqslant u \leqslant 1$, $-1 \leqslant v \leqslant 1$

8. $\mathbf{r}(u, v) = \langle u, v^3, -v \rangle$,
$-2 \leqslant u \leqslant 2$, $-2 \leqslant v \leqslant 2$

9. $\mathbf{r}(u, v) = \langle u^3, u\sin v, u\cos v \rangle$,
$-1 \leqslant u \leqslant 1$, $0 \leqslant v \leqslant 2\pi$

10. $\mathbf{r}(u, v) = \langle u, \sin(u + v), \sin v \rangle$,
$-\pi \leqslant u \leqslant \pi$, $-\pi \leqslant v \leqslant \pi$

11. $x = \sin v$, $\quad y = \cos u \sin 4v$, $\quad z = \sin 2u \sin 4v$,
$0 \leqslant u \leqslant 2\pi$, $-\pi/2 \leqslant v \leqslant \pi/2$

12. $x = \cos u$, $\quad y = \sin u \sin v$, $\quad z = \cos v$,
$0 \leqslant u \leqslant 2\pi$, $0 \leqslant v \leqslant 2\pi$

13–18 Match the equations with the graphs labeled I–VI and give reasons for your answers. Determine which families of grid curves have u constant and which have v constant.

13. $\mathbf{r}(u, v) = u\cos v\,\mathbf{i} + u\sin v\,\mathbf{j} + v\,\mathbf{k}$

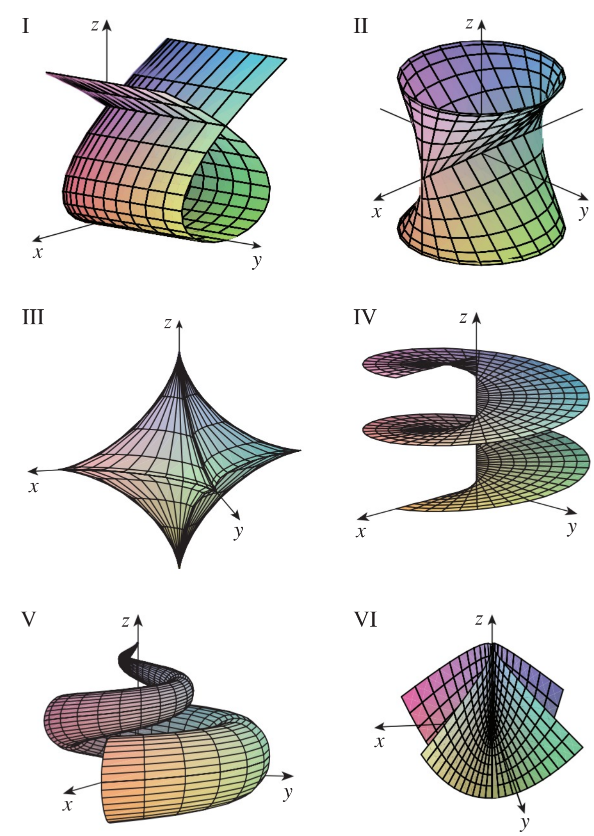
14. $\mathbf{r}(u, v) = uv^2\,\mathbf{i} + u^2v\,\mathbf{j} + (u^2 - v^2)\,\mathbf{k}$

15. $\mathbf{r}(u, v) = (u^3 - u)\,\mathbf{i} + v^2\,\mathbf{j} + u^2\,\mathbf{k}$

16. $x = (1 - u)(3 + \cos v)\cos 4\pi u$,
$y = (1 - u)(3 + \cos v)\sin 4\pi u$,
$z = 3u + (1 - u)\sin v$

17. $x = \cos^3 u \cos^3 v$, $\quad y = \sin^3 u \cos^3 v$, $\quad z = \sin^3 v$

18. $x = \sin u$, $\quad y = \cos u \sin v$, $\quad z = \sin v$



19–26 Find a parametric representation for the surface.

19. The plane through the origin that contains the vectors $\mathbf{i} - \mathbf{j}$ and $\mathbf{j} - \mathbf{k}$

20. The plane that passes through the point $(0, -1, 5)$ and contains the vectors $\langle 2, 1, 4 \rangle$ and $\langle -3, 2, 5 \rangle$

21. The part of the hyperboloid $4x^2 - 4y^2 - z^2 = 4$ that lies in front of the yz-plane

22. The part of the ellipsoid $x^2 + 2y^2 + 3z^2 = 1$ that lies to the left of the xz-plane

23. The part of the sphere $x^2 + y^2 + z^2 = 4$ that lies above the cone $z = \sqrt{x^2 + y^2}$

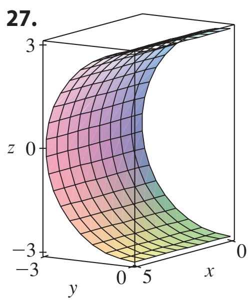
24. The part of the cylinder $x^2 + z^2 = 9$ that lies above the xy-plane and between the planes $y = -4$ and $y = 4$

25. The part of the sphere $x^2 + y^2 + z^2 = 36$ that lies between the planes $z = 0$ and $z = 3\sqrt{3}$

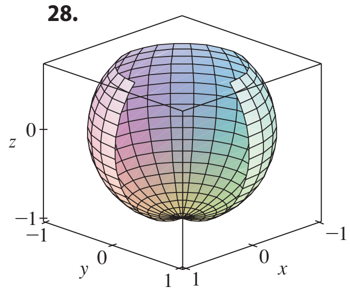
26. The part of the plane $z = x + 3$ that lies inside the cylinder $x^2 + y^2 = 1$

27–28 Use a computer to produce a graph that looks like the given one.

27.



28.



29. Find parametric equations for the surface obtained by rotating the curve $y = 1/(1 + x^2)$, $-2 \leqslant x \leqslant 2$, about the x-axis and use them to graph the surface.

30. Find parametric equations for the surface obtained by rotating the curve $x = 1/y$, $y \geqslant 1$, about the y-axis and use them to graph the surface.

31. (a) What happens to the spiral tube in Example 2 (see Figure 5) if we replace $\cos u$ by $\sin u$ and $\sin u$ by $\cos u$?
(b) What happens if we replace $\cos u$ by $\cos 2u$ and $\sin u$ by $\sin 2u$?

32. The surface with parametric equations

$$x = 2\cos\theta + r\cos(\theta/2)$$
$$y = 2\sin\theta + r\cos(\theta/2)$$
$$z = r\sin(\theta/2)$$

where $-\frac{1}{2} \leqslant r \leqslant \frac{1}{2}$ and $0 \leqslant \theta \leqslant 2\pi$, is called a **Möbius strip**. Graph this surface with several viewpoints. What is unusual about it?

33–36 Find an equation of the tangent plane to the given parametric surface at the specified point.

33. $x = u + v, \quad y = 3u^2, \quad z = u - v; \quad (2, 3, 0)$

34. $x = u^2 + 1, \quad y = v^3 + 1, \quad z = u + v; \quad (5, 2, 3)$

35. $\mathbf{r}(u, v) = u\cos v\,\mathbf{i} + u\sin v\,\mathbf{j} + v\,\mathbf{k}; \quad u = 1, \ v = \pi/3$

36. $\mathbf{r}(u, v) = \sin u\,\mathbf{i} + \cos u\sin v\,\mathbf{j} + \sin v\,\mathbf{k};$
$u = \pi/6, \ v = \pi/6$

37–38 Find an equation of the tangent plane to the given parametric surface at the specified point. Graph the surface and the tangent plane.

37. $\mathbf{r}(u, v) = u^2\,\mathbf{i} + 2u\sin v\,\mathbf{j} + u\cos v\,\mathbf{k}; \quad u = 1, \ v = 0$

38. $\mathbf{r}(u, v) = (1 - u^2 - v^2)\,\mathbf{i} - v\,\mathbf{j} - u\,\mathbf{k}; \quad (-1, -1, -1)$

39–50 Find the area of the surface.

39. The part of the plane $3x + 2y + z = 6$ that lies in the first octant

40. The part of the plane with vector equation $\mathbf{r}(u, v) = \langle u + v, 2 - 3u, 1 + u - v \rangle$ that is given by $0 \leqslant u \leqslant 2$, $-1 \leqslant v \leqslant 1$

41. The part of the plane $x + 2y + 3z = 1$ that lies inside the cylinder $x^2 + y^2 = 3$

42. The part of the cone $z = \sqrt{x^2 + y^2}$ that lies between the plane $y = x$ and the cylinder $y = x^2$

43. The surface $z = \frac{2}{3}(x^{3/2} + y^{3/2})$, $0 \leqslant x \leqslant 1$, $0 \leqslant y \leqslant 1$

44. The part of the surface $z = 4 - 2x^2 + y$ that lies above the triangle with vertices $(0, 0)$, $(1, 0)$, and $(1, 1)$

45. The part of the surface $z = xy$ that lies within the cylinder $x^2 + y^2 = 1$

46. The part of the surface $x = z^2 + y$ that lies between the planes $y = 0$, $y = 2$, $z = 0$, and $z = 2$

47. The part of the paraboloid $y = x^2 + z^2$ that lies within the cylinder $x^2 + z^2 = 16$

48. The helicoid (or spiral ramp) with vector equation $\mathbf{r}(u, v) = u\cos v\,\mathbf{i} + u\sin v\,\mathbf{j} + v\,\mathbf{k}$, $0 \leqslant u \leqslant 1$, $0 \leqslant v \leqslant \pi$

49. The surface with parametric equations $x = u^2$, $y = uv$, $z = \frac{1}{2}v^2$, $0 \leqslant u \leqslant 1$, $0 \leqslant v \leqslant 2$

50. The part of the sphere $x^2 + y^2 + z^2 = b^2$ that lies inside the cylinder $x^2 + y^2 = a^2$, where $0 < a < b$

51. If the equation of a surface S is $z = f(x, y)$, where $x^2 + y^2 \leqslant R^2$, and you know that $|f_x| \leqslant 1$ and $|f_y| \leqslant 1$, what can you say about $A(S)$?

52–53 Find the area of the surface correct to four decimal places by first simplifying an expression for area to one in terms of a single integral and then evaluating the integral numerically.

52. The part of the surface $z = \cos(x^2 + y^2)$ that lies inside the cylinder $x^2 + y^2 = 1$

53. The part of the surface $z = \ln(x^2 + y^2 + 2)$ that lies above the disk $x^2 + y^2 \leqslant 1$

54. Use a computer algebra system to find, to four decimal places, the area of the part of the surface $z = (1 + x^2)/(1 + y^2)$ that lies above the square $|x| + |y| \leqslant 1$. Illustrate by graphing this part of the surface.

55. (a) Use the Midpoint Rule for double integrals (see Section 15.1) with six squares to estimate the area of the surface $z = 1/(1 + x^2 + y^2)$, $0 \leqslant x \leqslant 6$, $0 \leqslant y \leqslant 4$.

(b) Use a computer algebra system to approximate the surface area in part (a) to four decimal places. Compare with the answer to part (a).

56. Use a computer algebra system to find the area of the surface with vector equation

$$\mathbf{r}(u, v) = \langle \cos^3 u \cos^3 v, \sin^3 u \cos^3 v, \sin^3 v \rangle$$

$0 \leqslant u \leqslant \pi$, $0 \leqslant v \leqslant 2\pi$. State your answer correct to four decimal places.

57. Use a computer algebra system to find the exact area of the surface $z = 1 + 2x + 3y + 4y^2$, $1 \leqslant x \leqslant 4$, $0 \leqslant y \leqslant 1$.

58. (a) Set up, but do not evaluate, a double integral for the area of the surface with parametric equations $x = au \cos v$, $y = bu \sin v$, $z = u^2$, $0 \leqslant u \leqslant 2$, $0 \leqslant v \leqslant 2\pi$.

(b) Eliminate the parameters to show that the surface is an elliptic paraboloid and set up another double integral for the surface area.

(c) Use the parametric equations in part (a) with $a = 2$ and $b = 3$ to graph the surface.

(d) For the case $a = 2$, $b = 3$, use a computer algebra system to find the surface area correct to four decimal places.

59. (a) Show that the parametric equations $x = a \sin u \cos v$, $y = b \sin u \sin v$, $z = c \cos u$, $0 \leqslant u \leqslant \pi$, $0 \leqslant v \leqslant 2\pi$, represent an ellipsoid.

(b) Use the parametric equations in part (a) to graph the ellipsoid for the case $a = 1$, $b = 2$, $c = 3$.

(c) Set up, but do not evaluate, a double integral for the surface area of the ellipsoid in part (b).

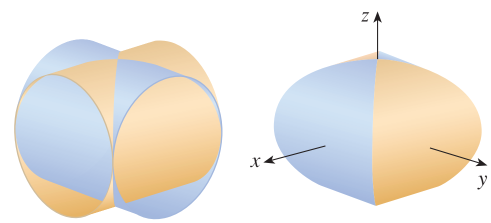
60. (a) Show that the parametric equations $x = a \cosh u \cos v$, $y = b \cosh u \sin v$, $z = c \sinh u$, represent a hyperboloid of one sheet.

(b) Use the parametric equations in part (a) to graph the hyperboloid for the case $a = 1$, $b = 2$, $c = 3$.

(c) Set up, but do not evaluate, a double integral for the surface area of the part of the hyperboloid in part (b) that lies between the planes $z = -3$ and $z = 3$.

61. Find the area of the part of the sphere $x^2 + y^2 + z^2 = 4z$ that lies inside the paraboloid $z = x^2 + y^2$.

62. The figure shows the surface created when the cylinder $y^2 + z^2 = 1$ intersects the cylinder $x^2 + z^2 = 1$. Find the area of this surface.

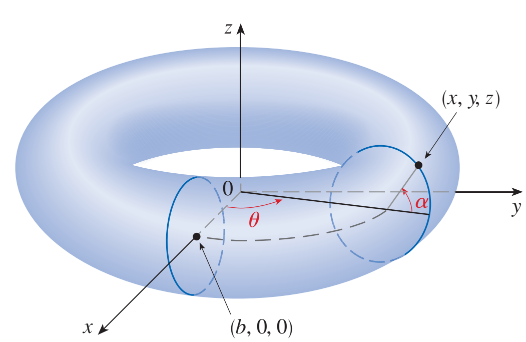


63. Find the area of the part of the sphere $x^2 + y^2 + z^2 = a^2$ that lies inside the cylinder $x^2 + y^2 = ax$.

64. (a) Find a parametric representation for the torus obtained by rotating about the z-axis the circle in the xz-plane with center $(b, 0, 0)$ and radius $a < b$. [*Hint:* Take as parameters the angles θ and α shown in the figure.]

(b) Use the parametric equations found in part (a) to graph the torus for several values of a and b.

(c) Use the parametric representation from part (a) to find the surface area of the torus.



16.7 Surface Integrals

The relationship between surface integrals and surface area is much the same as the relationship between line integrals and arc length. Suppose f is a function of three variables whose domain includes a surface S. We will define the surface integral of f over S in such a way that, in the case where $f(x, y, z) = 1$, the value of the surface integral is equal where ε_0 is a constant (called the permittivity of free space) that depends on the units used. (In the SI system, $\varepsilon_0 \approx 8.8542 \times 10^{-12}\ \mathrm{C^2/N\cdot m^2}$.) Therefore, if the vector field $\mathbf{F}$ in Example 4 represents an electric field, we can conclude that the charge enclosed by S is $Q = \frac{4}{3}\pi\varepsilon_0$.

Another application of surface integrals occurs in the study of heat flow. Suppose the temperature at a point (x, y, z) in a body is $u(x, y, z)$. Then the **heat flow** is defined as the vector field

$$\mathbf{F} = -K\,\nabla u$$

where K is an experimentally determined constant called the **conductivity** of the substance. The rate of heat flow across the surface S in the body is then given by the surface integral

$$\iint_S \mathbf{F}\cdot d\mathbf{S} = -K\iint_S \nabla u \cdot d\mathbf{S}$$

EXAMPLE 6 The temperature u in a metal ball is proportional to the square of the distance from the center of the ball. Find the rate of heat flow across a sphere S of radius a with center at the center of the ball.

SOLUTION Taking the center of the ball to be at the origin, we have

$$u(x, y, z) = C(x^2 + y^2 + z^2)$$

where C is the proportionality constant. Then the heat flow is

$$\mathbf{F}(x, y, z) = -K\,\nabla u = -KC(2x\,\mathbf{i} + 2y\,\mathbf{j} + 2z\,\mathbf{k})$$

where K is the conductivity of the metal. Instead of using the usual parametrization of the sphere as in Example 4, we observe that the outward unit normal to the sphere $x^2 + y^2 + z^2 = a^2$ at the point (x, y, z) is

$$\mathbf{n} = \frac{1}{a}(x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k})$$

and so

$$\mathbf{F}\cdot\mathbf{n} = -\frac{2KC}{a}(x^2 + y^2 + z^2)$$

But on S we have $x^2 + y^2 + z^2 = a^2$, so $\mathbf{F}\cdot\mathbf{n} = -2aKC$. Therefore the rate of heat flow across S is

$$\iint_S \mathbf{F}\cdot d\mathbf{S} = \iint_S \mathbf{F}\cdot\mathbf{n}\,dS = -2aKC\iint_S dS$$

$$= -2aKCA(S) = -2aKC(4\pi a^2) = -8KC\pi a^3$$

■

16.7 Exercises

1. Let S be the surface of the box enclosed by the planes $x = \pm 1$, $y = \pm 1$, $z = \pm 1$. Approximate $\iint_S \cos(x + 2y + 3z)\,dS$ by using a Riemann sum as in Definition 1, taking the patches S_{ij} to be the squares that are the faces of the box S and the points P_{ij}^* to be the centers of the squares.

2. A surface S consists of the cylinder $x^2 + y^2 = 1$, $-1 \leqslant z \leqslant 1$, together with its top and bottom disks. Suppose you know that f is a continuous function with

$$f(\pm 1, 0, 0) = 2 \qquad f(0, \pm 1, 0) = 3 \qquad f(0, 0, \pm 1) = 4$$

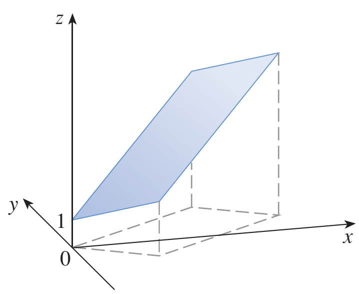
Estimate the value of $\iint_S f(x, y, z)\, dS$ by using a Riemann sum, taking the patches S_{ij} to be four quarter-cylinders and the top and bottom disks.

3. Let H be the hemisphere $x^2 + y^2 + z^2 = 50$, $z \geqslant 0$, and suppose f is a continuous function with $f(3, 4, 5) = 7$, $f(3, -4, 5) = 8$, $f(-3, 4, 5) = 9$, and $f(-3, -4, 5) = 12$. By dividing H into four patches, estimate the value of $\iint_H f(x, y, z)\, dS$.

4. Suppose that $f(x, y, z) = g\left(\sqrt{x^2 + y^2 + z^2}\right)$, where g is a function of one variable such that $g(2) = -5$. Evaluate $\iint_S f(x, y, z)\, dS$, where S is the sphere $x^2 + y^2 + z^2 = 4$.

5–20 Evaluate the surface integral.

5. $\iint_S (x + y + z)\, dS$,
S is the parallelogram with parametric equations $x = u + v$, $y = u - v$, $z = 1 + 2u + v$, $0 \leqslant u \leqslant 2$, $0 \leqslant v \leqslant 1$



6. $\iint_S xyz\, dS$,
S is the cone with parametric equations $x = u \cos v$, $y = u \sin v$, $z = u$, $0 \leqslant u \leqslant 1$, $0 \leqslant v \leqslant \pi/2$

7. $\iint_S y\, dS$, S is the helicoid with vector equation $\mathbf{r}(u, v) = \langle u \cos v, u \sin v, v \rangle$, $0 \leqslant u \leqslant 1$, $0 \leqslant v \leqslant \pi$

8. $\iint_S (x^2 + y^2)\, dS$,
S is the surface with vector equation $\mathbf{r}(u, v) = \langle 2uv, u^2 - v^2, u^2 + v^2 \rangle$, $u^2 + v^2 \leqslant 1$

9. $\iint_S x^2yz\, dS$, S is the part of the plane $z = 1 + 2x + 3y$ that lies above the rectangle $[0, 3] \times [0, 2]$

10. $\iint_S xz\, dS$, S is the part of the plane $2x + 2y + z = 4$ that lies in the first octant

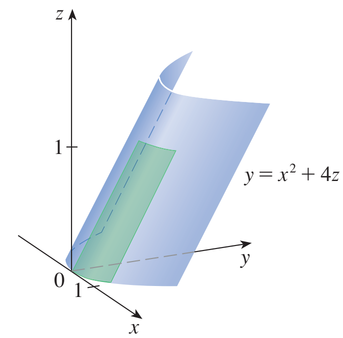
11. $\iint_S x\, dS$,
S is the triangular region with vertices $(1, 0, 0)$, $(0, -2, 0)$, and $(0, 0, 4)$

12. $\iint_S y\, dS$,
S is the surface $z = \frac{2}{3}(x^{3/2} + y^{3/2})$, $0 \leqslant x \leqslant 1$, $0 \leqslant y \leqslant 1$

13. $\iint_S z^2\, dS$,
S is the part of the paraboloid $x = y^2 + z^2$ given by $0 \leqslant x \leqslant 1$

14. $\iint_S y^2z^2\, dS$,
S is the part of the cone $y = \sqrt{x^2 + z^2}$ given by $0 \leqslant y \leqslant 5$

15. $\iint_S x\, dS$,
S is the surface $y = x^2 + 4z$, $0 \leqslant x \leqslant 1$, $0 \leqslant z \leqslant 1$

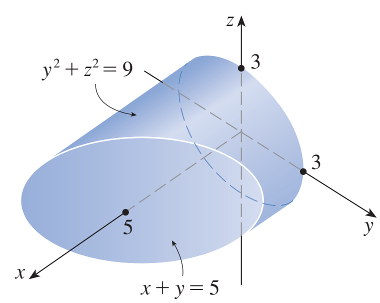


16. $\iint_S y^2\, dS$,
S is the part of the sphere $x^2 + y^2 + z^2 = 1$ that lies above the cone $z = \sqrt{x^2 + y^2}$

17. $\iint_S (x^2z + y^2z)\, dS$,
S is the hemisphere $x^2 + y^2 + z^2 = 4$, $z \geqslant 0$

18. $\iint_S (x + y + z)\, dS$,
S is the part of the half-cylinder $x^2 + z^2 = 1$, $z \geqslant 0$, that lies between the planes $y = 0$ and $y = 2$

19. $\iint_S xz\, dS$,
S is the boundary of the region enclosed by the cylinder $y^2 + z^2 = 9$ and the planes $x = 0$ and $x + y = 5$



20. $\iint_S (x^2 + y^2 + z^2)\, dS$,
S is the part of the cylinder $x^2 + y^2 = 9$ between the planes $z = 0$ and $z = 2$, together with its top and bottom disks

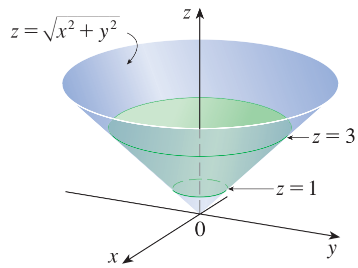
21–32 Evaluate the surface integral $\iint_S \mathbf{F} \cdot d\mathbf{S}$ for the given vector field $\mathbf{F}$ and the oriented surface S. In other words, find the flux of $\mathbf{F}$ across S. For closed surfaces, use the positive (outward) orientation.

21. $\mathbf{F}(x, y, z) = ze^{xy}\, \mathbf{i} - 3ze^{xy}\, \mathbf{j} + xy\, \mathbf{k}$,
S is the parallelogram of Exercise 5 with upward orientation

22. $\mathbf{F}(x, y, z) = z\, \mathbf{i} + y\, \mathbf{j} + x\, \mathbf{k}$,
S is the helicoid of Exercise 7 with upward orientation

23. $\mathbf{F}(x, y, z) = xy\, \mathbf{i} + yz\, \mathbf{j} + zx\, \mathbf{k}$, S is the part of the paraboloid $z = 4 - x^2 - y^2$ that lies above the square $0 \leqslant x \leqslant 1$, $0 \leqslant y \leqslant 1$, and has upward orientation

24. $\mathbf{F}(x, y, z) = -x\,\mathbf{i} - y\,\mathbf{j} + z^3\,\mathbf{k}$, S is the part of the cone $z = \sqrt{x^2 + y^2}$ between the planes $z = 1$ and $z = 3$ with downward orientation



25. $\mathbf{F}(x, y, z) = x\,\mathbf{i} + y\,\mathbf{j} + z^2\,\mathbf{k}$, S is the sphere with radius 1 and center the origin

26. $\mathbf{F}(x, y, z) = y\,\mathbf{i} - x\,\mathbf{j} + 2z\,\mathbf{k}$, S is the hemisphere $x^2 + y^2 + z^2 = 4$, $z \geqslant 0$, oriented downward

27. $\mathbf{F}(x, y, z) = y\,\mathbf{j} - z\,\mathbf{k}$,
S consists of the paraboloid $y = x^2 + z^2$, $0 \leqslant y \leqslant 1$, and the disk $x^2 + z^2 \leqslant 1$, $y = 1$

28. $\mathbf{F}(x, y, z) = yz\,\mathbf{i} + zx\,\mathbf{j} + xy\,\mathbf{k}$, S is the surface $z = x \sin y$, $0 \leqslant x \leqslant 2$, $0 \leqslant y \leqslant \pi$, with upward orientation

29. $\mathbf{F}(x, y, z) = x\,\mathbf{i} + 2y\,\mathbf{j} + 3z\,\mathbf{k}$,
S is the cube with vertices $(\pm 1, \pm 1, \pm 1)$

30. $\mathbf{F}(x, y, z) = x\,\mathbf{i} + y\,\mathbf{j} + 5\,\mathbf{k}$, S is the boundary of the region enclosed by the cylinder $x^2 + z^2 = 1$ and the planes $y = 0$ and $x + y = 2$

31. $\mathbf{F}(x, y, z) = x^2\,\mathbf{i} + y^2\mathbf{j} + z^2\,\mathbf{k}$, S is the boundary of the solid half-cylinder $0 \leqslant z \leqslant \sqrt{1 - y^2}$, $0 \leqslant x \leqslant 2$

32. $\mathbf{F}(x, y, z) = y\,\mathbf{i} + (z - y)\,\mathbf{j} + x\,\mathbf{k}$,
S is the surface of the tetrahedron with vertices $(0, 0, 0)$, $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$

T **33.** Use a computer algebra system to evaluate $\iint_S (x^2 + y^2 + z^2)\,dS$ correct to four decimal places, where S is the surface $z = xe^y$, $0 \leqslant x \leqslant 1$, $0 \leqslant y \leqslant 1$.

T **34.** Use a computer algebra system to find the exact value of $\iint_S xyz\,dS$, where S is the surface $z = x^2y^2$, $0 \leqslant x \leqslant 1$, $0 \leqslant y \leqslant 2$.

T **35.** Use a computer algebra system to find the value of $\iint_S x^2y^2z^2\,dS$ correct to four decimal places, where S is the part of the paraboloid $z = 3 - 2x^2 - y^2$ that lies above the xy-plane.

T **36.** Use a computer algebra system to find the flux of

$$\mathbf{F}(x, y, z) = \sin(xyz)\,\mathbf{i} + x^2y\,\mathbf{j} + z^2e^{x/5}\,\mathbf{k}$$

across the part of the cylinder $4y^2 + z^2 = 4$ that lies above the xy-plane and between the planes $x = -2$ and $x = 2$ with upward orientation. Illustrate by graphing the cylinder and the vector field on the same screen.

37. Find a formula for $\iint_S \mathbf{F} \cdot d\mathbf{S}$ similar to Formula 10 for the case where S is given by $y = h(x, z)$ and $\mathbf{n}$ is the unit normal that points toward the left (when the axes are drawn in the usual way).

38. Find a formula for $\iint_S \mathbf{F} \cdot d\mathbf{S}$ similar to Formula 10 for the case where S is given by $x = k(y, z)$ and $\mathbf{n}$ is the unit normal that points forward (that is, toward the viewer when the axes are drawn in the usual way).

39. Find the center of mass of the hemisphere $x^2 + y^2 + z^2 = a^2$, $z \geqslant 0$, if it has constant density.

40. Find the mass of a thin funnel in the shape of a cone $z = \sqrt{x^2 + y^2}$, $1 \leqslant z \leqslant 4$, if its density function is $\rho(x, y, z) = 10 - z$.

41. (a) Give an integral expression for the moment of inertia I_z about the z-axis of a thin sheet in the shape of a surface S if the density function is ρ.
(b) Find the moment of inertia about the z-axis of the funnel in Exercise 40.

42. Let S be the part of the sphere $x^2 + y^2 + z^2 = 25$ that lies above the plane $z = 4$. If S has constant density k, find (a) the center of mass and (b) the moment of inertia about the z-axis.

43. A fluid has density 870 kg/m^3 and flows with velocity $\mathbf{v} = z\,\mathbf{i} + y^2\,\mathbf{j} + x^2\,\mathbf{k}$, where x, y, and z are measured in meters and the components of $\mathbf{v}$ in meters per second. Find the rate of flow outward through the cylinder $x^2 + y^2 = 4$, $0 \leqslant z \leqslant 1$.

44. Seawater has density 1025 kg/m^3 and flows in a velocity field $\mathbf{v} = y\,\mathbf{i} + x\,\mathbf{j}$, where x, y, and z are measured in meters and the components of $\mathbf{v}$ in meters per second. Find the rate of flow outward through the hemisphere $x^2 + y^2 + z^2 = 9$, $z \geqslant 0$.

45. Use Gauss's Law to find the charge contained in the solid hemisphere $x^2 + y^2 + z^2 \leqslant a^2$, $z \geqslant 0$, if the electric field is

$$\mathbf{E}(x, y, z) = x\,\mathbf{i} + y\,\mathbf{j} + 2z\,\mathbf{k}$$

46. Use Gauss's Law to find the charge enclosed by the cube with vertices $(\pm 1, \pm 1, \pm 1)$ if the electric field is

$$\mathbf{E}(x, y, z) = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$$

47. The temperature at the point (x, y, z) in a substance with conductivity $K = 6.5$ is $u(x, y, z) = 2y^2 + 2z^2$. Find the rate of heat flow inward across the cylindrical surface $y^2 + z^2 = 6$, $0 \leqslant x \leqslant 4$.

48. The temperature at a point in a ball with conductivity K is inversely proportional to the distance from the center of the ball. Find the rate of heat flow across a sphere S of radius a with center at the center of the ball.

49. Let $\mathbf{F}$ be an inverse square field, that is, $\mathbf{F}(\mathbf{r}) = c\mathbf{r}/|\,\mathbf{r}\,|^3$ for some constant c, where $\mathbf{r} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$. Show that the flux of $\mathbf{F}$ across a sphere S with center the origin is independent of the radius of S.

Now let $P_0(x_0, y_0, z_0)$ be a point in the fluid and let S_a be a small disk with radius a and center P_0. Then $(\text{curl } \mathbf{F})(P) \approx (\text{curl } \mathbf{F})(P_0)$ for all points P on S_a because curl $\mathbf{F}$ is continuous. Thus, by Stokes' Theorem, we get the following approximation to the circulation around the boundary circle C_a:

$$\int_{C_a} \mathbf{v} \cdot d\mathbf{r} = \iint_{S_a} \text{curl } \mathbf{v} \cdot d\mathbf{S} = \iint_{S_a} \text{curl } \mathbf{v} \cdot \mathbf{n} \, dS$$

$$\approx \iint_{S_a} \text{curl } \mathbf{v}(P_0) \cdot \mathbf{n}(P_0) \, dS = \text{curl } \mathbf{v}(P_0) \cdot \mathbf{n}(P_0) \pi a^2$$

Imagine a tiny paddle wheel placed in the fluid at a point P, as in Figure 7; the paddle wheel rotates fastest when its axis is parallel to curl $\mathbf{v}$.

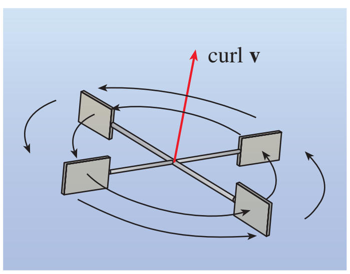


FIGURE 7

This approximation becomes better as $a \to 0$ and we have

$$\boxed{4} \qquad \text{curl } \mathbf{v}(P_0) \cdot \mathbf{n}(P_0) = \lim_{a \to 0} \frac{1}{\pi a^2} \int_{C_a} \mathbf{v} \cdot d\mathbf{r}$$

Equation 4 gives the relationship between the curl and the circulation. It shows that curl $\mathbf{v} \cdot \mathbf{n}$ is a measure of the rotating effect of the fluid about the axis $\mathbf{n}$. The curling effect is greatest about the axis parallel to curl $\mathbf{v}$.

Finally, we mention that Stokes' Theorem can be used to prove Theorem 16.5.4 (which states that if curl $\mathbf{F} = \mathbf{0}$ on all of $\mathbb{R}^3$, then $\mathbf{F}$ is conservative). From our previous work (Theorems 16.3.3 and 16.3.4), we know that $\mathbf{F}$ is conservative if $\int_C \mathbf{F} \cdot d\mathbf{r} = 0$ for every closed path C. Given C, suppose we can find an orientable surface S whose boundary is C. (This can be done, but the proof requires advanced techniques.) Then Stokes' Theorem gives

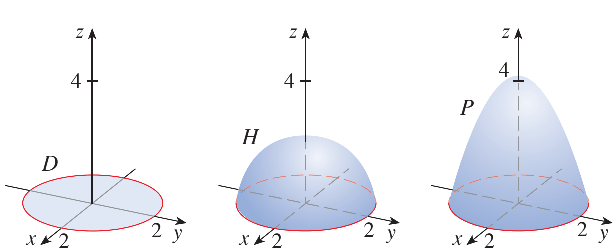
$$\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \text{curl } \mathbf{F} \cdot d\mathbf{S} = \iint_S \mathbf{0} \cdot d\mathbf{S} = 0$$

A curve that is not simple can be broken into a number of simple curves, and the integrals around these simple curves are all 0. Adding these integrals, we obtain $\int_C \mathbf{F} \cdot d\mathbf{r} = 0$ for any closed curve C.

16.8 Exercises

1. A disk D, a hemisphere H, and a portion P of a paraboloid are shown. Suppose $\mathbf{F}$ is a vector field on $\mathbb{R}^3$ whose components have continuous partial derivatives. Explain why this statement is true:

$$\iint_D \text{curl } \mathbf{F} \cdot d\mathbf{S} = \iint_H \text{curl } \mathbf{F} \cdot d\mathbf{S} = \iint_P \text{curl } \mathbf{F} \cdot d\mathbf{S}$$



2–6 Use Stokes' Theorem to evaluate $\iint_S \text{curl } \mathbf{F} \cdot d\mathbf{S}$.

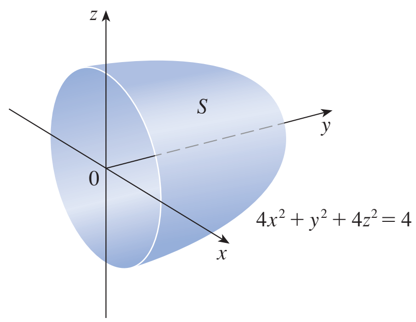
2. $\mathbf{F}(x, y, z) = x^2 \sin z \, \mathbf{i} + y^2 \, \mathbf{j} + xy \, \mathbf{k}$,
S is the part of the paraboloid $z = 1 - x^2 - y^2$ that lies above the xy-plane, oriented upward

3. $\mathbf{F}(x, y, z) = ze^y \, \mathbf{i} + x \cos y \, \mathbf{j} + xz \sin y \, \mathbf{k}$,
S is the hemisphere $x^2 + y^2 + z^2 = 16$, $y \geqslant 0$, oriented in the direction of the positive y-axis

4. $\mathbf{F}(x, y, z) = \tan^{-1}(x^2 yz^2) \, \mathbf{i} + x^2 y \, \mathbf{j} + x^2 z^2 \, \mathbf{k}$,
S is the cone $x = \sqrt{y^2 + z^2}$, $0 \leqslant x \leqslant 2$, oriented in the direction of the positive x-axis

5. $\mathbf{F}(x, y, z) = xyz \, \mathbf{i} + xy \, \mathbf{j} + x^2 yz \, \mathbf{k}$,
S consists of the top and the four sides (but not the bottom) of the cube with vertices $(\pm 1, \pm 1, \pm 1)$, oriented outward

6. $\mathbf{F}(x, y, z) = e^{xy}\,\mathbf{i} + e^{xz}\,\mathbf{j} + x^2 z\,\mathbf{k}$,
S is the half of the ellipsoid $4x^2 + y^2 + 4z^2 = 4$ that lies to the right of the xz-plane, oriented in the direction of the positive y-axis

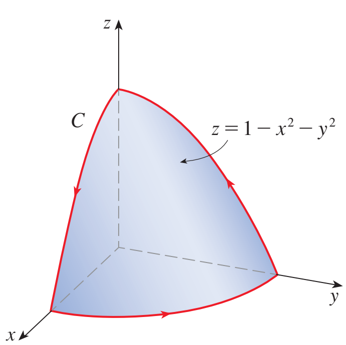


7–14 Use Stokes' Theorem to evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$. In each case C is oriented counterclockwise as viewed from above, unless otherwise stated.

7. $\mathbf{F}(x, y, z) = (x + y^2)\,\mathbf{i} + (y + z^2)\,\mathbf{j} + (z + x^2)\,\mathbf{k}$,
C is the triangle with vertices (1, 0, 0), (0, 1, 0), and (0, 0, 1)

8. $\mathbf{F}(x, y, z) = \mathbf{i} + (x + yz)\,\mathbf{j} + (xy - \sqrt{z})\,\mathbf{k}$,
C is the boundary of the part of the plane $3x + 2y + z = 1$ in the first octant

9. $\mathbf{F}(x, y, z) = xy\,\mathbf{i} + yz\,\mathbf{j} + zx\,\mathbf{k}$,
C is the boundary of the part of the paraboloid $z = 1 - x^2 - y^2$ in the first octant

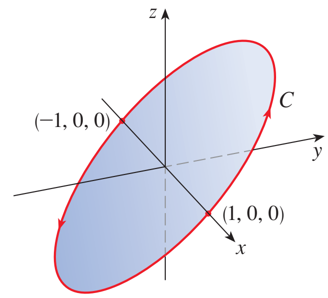


10. $\mathbf{F}(x, y, z) = 2y\,\mathbf{i} + xz\,\mathbf{j} + (x + y)\,\mathbf{k}$,
C is the curve of intersection of the plane $z = y + 2$ and the cylinder $x^2 + y^2 = 1$

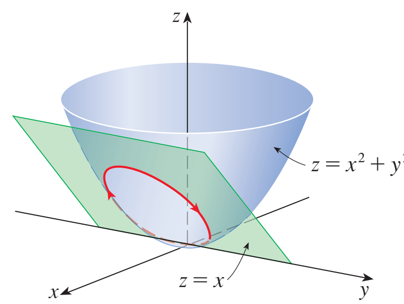
11. $\mathbf{F}(x, y, z) = \langle -yx^2, xy^2, e^{xy} \rangle$, C is the circle in the xy-plane of radius 2 centered at the origin

12. $\mathbf{F}(x, y, z) = ze^x\,\mathbf{i} + (z - y^3)\,\mathbf{j} + (x - z^3)\,\mathbf{k}$,
C is the circle $y^2 + z^2 = 4$, $x = 3$, oriented clockwise as viewed from the origin

13. $\mathbf{F}(x, y, z) = x^2 y\,\mathbf{i} + x^3\,\mathbf{j} + e^z \tan^{-1} z\,\mathbf{k}$,
C is the curve with parametric equations $x = \cos t$, $y = \sin t$, $z = \sin t$, $0 \leqslant t \leqslant 2\pi$



14. $\mathbf{F}(x, y, z) = \langle x^3 - z, xy, y + z^2 \rangle$, C is the curve of intersection of the paraboloid $z = x^2 + y^2$ and the plane $z = x$



15. (a) Use Stokes' Theorem to evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$, where

$$\mathbf{F}(x, y, z) = x^2 z\,\mathbf{i} + xy^2\,\mathbf{j} + z^2\,\mathbf{k}$$

and C is the curve of intersection of the plane $x + y + z = 1$ and the cylinder $x^2 + y^2 = 9$, oriented counterclockwise as viewed from above.

(b) Graph both the plane and the cylinder with domains chosen so that you can see the curve C and the surface that you used in part (a).

(c) Find parametric equations for C and use them to graph C.

16. (a) Use Stokes' Theorem to evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F}(x, y, z) = x^2 y\,\mathbf{i} + \frac{1}{3}x^3\,\mathbf{j} + xy\,\mathbf{k}$ and C is the curve of intersection of the hyperbolic paraboloid $z = y^2 - x^2$ and the cylinder $x^2 + y^2 = 1$, oriented counterclockwise as viewed from above.

(b) Graph both the hyperbolic paraboloid and the cylinder with domains chosen so that you can see the curve C and the surface that you used in part (a).

(c) Find parametric equations for C and use them to graph C.

17–19 Verify that Stokes' Theorem is true for the given vector field $\mathbf{F}$ and surface S.

17. $\mathbf{F}(x, y, z) = -y\,\mathbf{i} + x\,\mathbf{j} - 2\,\mathbf{k}$,
S is the cone $z^2 = x^2 + y^2$, $0 \leqslant z \leqslant 4$, oriented downward

18. $\mathbf{F}(x, y, z) = -2yz\,\mathbf{i} + y\,\mathbf{j} + 3x\,\mathbf{k}$,
S is the part of the paraboloid $z = 5 - x^2 - y^2$ that lies above the plane $z = 1$, oriented upward

19. $\mathbf{F}(x, y, z) = y\,\mathbf{i} + z\,\mathbf{j} + x\,\mathbf{k}$,
S is the hemisphere $x^2 + y^2 + z^2 = 1$, $y \geqslant 0$, oriented in the direction of the positive y-axis

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20. Let C be a simple closed smooth curve that lies in the plane $x + y + z = 1$. Show that the line integral

$$\int_C z\,dx - 2x\,dy + 3y\,dz$$

depends only on the area of the region enclosed by C and not on the shape of C or its location in the plane.

21. A particle moves along line segments from the origin to the points $(1, 0, 0)$, $(1, 2, 1)$, $(0, 2, 1)$, and back to the origin under the influence of the force field

$$\mathbf{F}(x, y, z) = z^2\,\mathbf{i} + 2xy\,\mathbf{j} + 4y^2\,\mathbf{k}$$

Find the work done.

22. Evaluate

$$\int_C (y + \sin x)\,dx + (z^2 + \cos y)\,dy + x^3\,dz$$

where C is the curve $\mathbf{r}(t) = \langle \sin t, \cos t, \sin 2t \rangle$, $0 \leqslant t \leqslant 2\pi$. [*Hint:* Observe that C lies on the surface $z = 2xy$.]

23. If S is a sphere and $\mathbf{F}$ satisfies the hypotheses of Stokes' Theorem, show that $\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = 0$.

24. Suppose S and C satisfy the hypotheses of Stokes' Theorem and f, g have continuous second-order partial derivatives. Use Exercises 26 and 28 in Section 16.5 to show the following.
(a) $\int_C (f\,\nabla g) \cdot d\mathbf{r} = \iint_S (\nabla f \times \nabla g) \cdot d\mathbf{S}$
(b) $\int_C (f\,\nabla f) \cdot d\mathbf{r} = 0$
(c) $\int_C (f\,\nabla g + g\,\nabla f) \cdot d\mathbf{r} = 0$

16.9 The Divergence Theorem

In Section 16.5 we rewrote Green's Theorem in a vector version as

$$\int_C \mathbf{F} \cdot \mathbf{n}\,ds = \iint_D \operatorname{div} \mathbf{F}(x, y)\,dA$$

where C is the positively oriented boundary curve of the plane region D. If we were seeking to extend this theorem to vector fields on $\mathbb{R}^3$, we might make the guess that

$$\boxed{1} \qquad \iint_S \mathbf{F} \cdot \mathbf{n}\,dS = \iiint_E \operatorname{div} \mathbf{F}(x, y, z)\,dV$$

where S is the boundary surface of the solid region E. It turns out that Equation 1 is true, under appropriate hypotheses, and is called the Divergence Theorem. Notice its similarity to Green's Theorem and Stokes' Theorem in that it relates the integral of a derivative of a function ($\operatorname{div} \mathbf{F}$ in this case) over a region to the integral of the original function $\mathbf{F}$ over the boundary of the region.

At this stage you may wish to review the various types of regions over which we were able to evaluate triple integrals in Section 15.6. We state and prove the Divergence Theorem for regions E that are simultaneously of types 1, 2, and 3 and we call such regions **simple solid regions**. (For instance, regions bounded by ellipsoids or rectangular boxes are simple solid regions.) The boundary of E is a closed surface, and we use the convention, introduced in Section 16.7, that the positive orientation is outward; that is, the unit normal vector $\mathbf{n}$ is directed outward from E.

The Divergence Theorem is sometimes called Gauss's Theorem after the great German mathematician Karl Friedrich Gauss (1777–1855), who discovered this theorem during his investigation of electrostatics. In Eastern Europe the Divergence Theorem is known as Ostrogradsky's Theorem after the Russian mathematician Mikhail Ostrogradsky (1801–1862), who published this result in 1826.

The Divergence Theorem Let E be a simple solid region and let S be the boundary surface of E, given with positive (outward) orientation. Let $\mathbf{F}$ be a vector field whose component functions have continuous partial derivatives on an open region that contains E. Then

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iiint_E \operatorname{div} \mathbf{F}\,dV$$

16.9 Exercises

1–4 Verify that the Divergence Theorem is true for the vector field $\mathbf{F}$ on the region E.

1. $\mathbf{F}(x, y, z) = 3x\,\mathbf{i} + xy\,\mathbf{j} + 2xz\,\mathbf{k}$,
E is the cube bounded by the planes $x = 0$, $x = 1$, $y = 0$, $y = 1$, $z = 0$, and $z = 1$

2. $\mathbf{F}(x, y, z) = y^2z^3\,\mathbf{i} + 2yz\,\mathbf{j} + 4z^2\,\mathbf{k}$,
E is the solid enclosed by the paraboloid $z = x^2 + y^2$ and the plane $z = 9$

3. $\mathbf{F}(x, y, z) = \langle z, y, x \rangle$,
E is the solid ball $x^2 + y^2 + z^2 \leqslant 16$

4. $\mathbf{F}(x, y, z) = \langle x^2, -y, z \rangle$,
E is the solid cylinder $y^2 + z^2 \leqslant 9$, $0 \leqslant x \leqslant 2$

5–17 Use the Divergence Theorem to calculate the surface integral $\iint_S \mathbf{F} \cdot d\mathbf{S}$; that is, calculate the flux of $\mathbf{F}$ across S.

5. $\mathbf{F}(x, y, z) = xye^z\,\mathbf{i} + xy^2z^3\,\mathbf{j} - ye^z\,\mathbf{k}$,
S is the surface of the box bounded by the coordinate planes and the planes $x = 3$, $y = 2$, and $z = 1$

6. $\mathbf{F}(x, y, z) = x^2yz\,\mathbf{i} + xy^2z\,\mathbf{j} + xyz^2\,\mathbf{k}$,
S is the surface of the box enclosed by the planes $x = 0$, $x = a$, $y = 0$, $y = b$, $z = 0$, and $z = c$, where a, b, and c are positive numbers

7. $\mathbf{F}(x, y, z) = 3xy^2\,\mathbf{i} + xe^z\,\mathbf{j} + z^3\,\mathbf{k}$,
S is the surface of the solid bounded by the cylinder $y^2 + z^2 = 1$ and the planes $x = -1$ and $x = 2$

8. $\mathbf{F}(x, y, z) = (x^3 + y^3)\,\mathbf{i} + (y^3 + z^3)\,\mathbf{j} + (z^3 + x^3)\,\mathbf{k}$,
S is the sphere with center the origin and radius 2

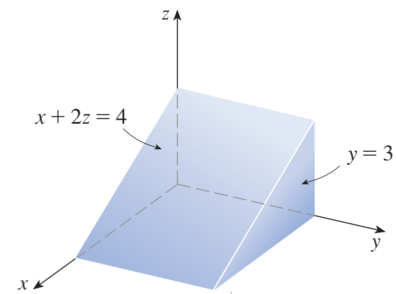
9. $\mathbf{F}(x, y, z) = xe^y\,\mathbf{i} + (z - e^y)\,\mathbf{j} - xy\,\mathbf{k}$,
S is the ellipsoid $x^2 + 2y^2 + 3z^2 = 4$

10. $\mathbf{F}(x, y, z) = e^y \tan z\,\mathbf{i} + x^2y\,\mathbf{j} + e^x \cos y\,\mathbf{k}$,
S is the surface of the solid that lies above the xy-plane and below the surface $z = 2 - x - y^3$, $-1 \leqslant x \leqslant 1$, $-1 \leqslant y \leqslant 1$

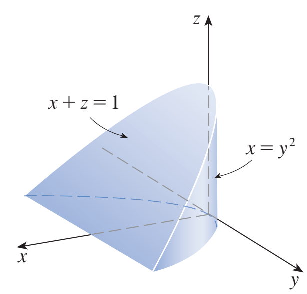
11. $\mathbf{F}(x, y, z) = (2x^3 + y^3)\,\mathbf{i} + (y^3 + z^3)\,\mathbf{j} + 3y^2z\,\mathbf{k}$,
S is the surface of the solid bounded by the paraboloid $z = 1 - x^2 - y^2$ and the xy-plane

12. $\mathbf{F}(x, y, z) = (xy + 2xz)\,\mathbf{i} + (x^2 + y^2)\,\mathbf{j} + (xy - z^2)\,\mathbf{k}$,
S is the surface of the solid bounded by the cylinder $x^2 + y^2 = 4$ and the planes $z = y - 2$ and $z = 0$

13. $\mathbf{F}(x, y, z) = x^2z\,\mathbf{i} + xz^3\,\mathbf{j} + y\ln(x + 1)\,\mathbf{k}$,
S is the surface of the solid bounded by the planes $x + 2z = 4$, $y = 3$, $x = 0$, $y = 0$, and $z = 0$



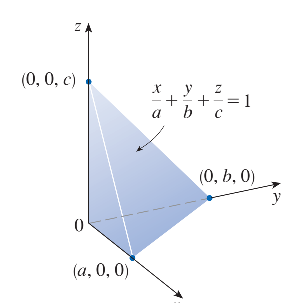
14. $\mathbf{F}(x, y, z) = (xy - z^2)\,\mathbf{i} + x^3\sqrt{z}\,\mathbf{j} + (xy + z^2)\,\mathbf{k}$,
S is the surface of the solid bounded by the cylinder $x = y^2$ and the planes $x + z = 1$ and $z = 0$



15. $\mathbf{F}(x, y, z) = z\,\mathbf{i} + y\,\mathbf{j} + zx\,\mathbf{k}$,
S is the surface of the tetrahedron enclosed by the coordinate planes and the plane

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$$

where a, b, and c are positive numbers



16. $\mathbf{F} = |\mathbf{r}|^2\mathbf{r}$, where $\mathbf{r} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$,
S is the sphere with radius R and center the origin

17. $\mathbf{F} = |\mathbf{r}|\,\mathbf{r}$, where $\mathbf{r} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$,
S consists of the hemisphere $z = \sqrt{1 - x^2 - y^2}$ and the disk $x^2 + y^2 \leqslant 1$ in the xy-plane

T **18.** Plot the vector field

$$\mathbf{F}(x, y, z) = \sin x \cos^2 y\,\mathbf{i} + \sin^3 y \cos^4 z\,\mathbf{j} + \sin^5 z \cos^6 x\,\mathbf{k}$$

in the cube cut from the first octant by the planes $x = \pi/2$, $y = \pi/2$, and $z = \pi/2$. Then use a computer algebra system to compute the flux across the surface of the cube.

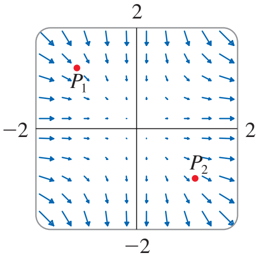
19. Use the Divergence Theorem to evaluate $\iint_S \mathbf{F} \cdot d\mathbf{S}$, where

$$\mathbf{F}(x, y, z) = z^2 x\,\mathbf{i} + \left(\tfrac{1}{3}y^3 + \tan^{-1}z\right)\mathbf{j} + (x^2 z + y^2)\,\mathbf{k}$$

and S is the top half of the sphere $x^2 + y^2 + z^2 = 1$. [*Hint:* Note that S is not a closed surface. First compute integrals over S_1 and S_2, where S_1 is the disk $x^2 + y^2 \leqslant 1$, oriented downward, and $S_2 = S \cup S_1$.]

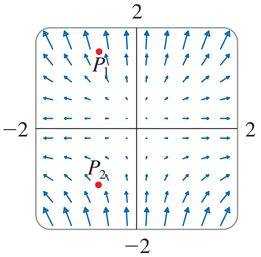
20. Let $\mathbf{F}(x, y, z) = z \tan^{-1}(y^2)\,\mathbf{i} + z^3 \ln(x^2 + 1)\,\mathbf{j} + z\,\mathbf{k}$. Find the flux of $\mathbf{F}$ across the part of the paraboloid $x^2 + y^2 + z = 2$ that lies above the plane $z = 1$ and is oriented upward.

21. A vector field $\mathbf{F}$ is shown. Use the interpretation of divergence derived in this section to determine whether the points P_1 and P_2 are sources or sinks.



22. (a) Are the points P_1 and P_2 sources or sinks for the vector field $\mathbf{F}$ shown in the figure? Give an explanation based solely on the picture.

(b) Given that $\mathbf{F}(x, y) = \langle x, y^2 \rangle$, use the definition of divergence to verify your answer to part (a).



23–24 Plot the vector field and guess where div $\mathbf{F} > 0$ and where div $\mathbf{F} < 0$. Then calculate div $\mathbf{F}$ to check your guess.

23. $\mathbf{F}(x, y) = \langle xy, x + y^2 \rangle$ **24.** $\mathbf{F}(x, y) = \langle x^2, y^2 \rangle$

25. Verify that div $\mathbf{E} = 0$ for the electric field $\mathbf{E}(\mathbf{x}) = \dfrac{\varepsilon Q}{|\mathbf{x}|^3}\,\mathbf{x}$.

26. Use the Divergence Theorem to evaluate

$$\iint_S (2x + 2y + z^2)\, dS$$

where S is the sphere $x^2 + y^2 + z^2 = 1$.

27–32 Prove each identity, assuming that S and E satisfy the conditions of the Divergence Theorem and the scalar functions and components of the vector fields have continuous second-order partial derivatives.

27. $\displaystyle\iint_S \mathbf{a} \cdot \mathbf{n}\, dS = 0$, where $\mathbf{a}$ is a constant vector

28. $\displaystyle V(E) = \frac{1}{3}\iint_S \mathbf{F} \cdot d\mathbf{S}$, where $\mathbf{F}(x, y, z) = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$

29. $\displaystyle\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = 0$ **30.** $\displaystyle\iint_S D_{\mathbf{n}} f\, dS = \iiint_E \nabla^2 f\, dV$

31. $\displaystyle\iint_S (f\nabla g) \cdot \mathbf{n}\, dS = \iiint_E (f\nabla^2 g + \nabla f \cdot \nabla g)\, dV$

32. $\displaystyle\iint_S (f\nabla g - g\nabla f) \cdot \mathbf{n}\, dS = \iiint_E (f\nabla^2 g - g\nabla^2 f)\, dV$

33. Suppose S and E satisfy the conditions of the Divergence Theorem and f is a scalar function with continuous partial derivatives. Prove that

$$\iint_S f\mathbf{n}\, dS = \iiint_E \nabla f\, dV$$

These surface and triple integrals of vector functions are vectors defined by integrating each component function. [*Hint:* Start by applying the Divergence Theorem to $\mathbf{F} = f\mathbf{c}$, where $\mathbf{c}$ is an arbitrary constant vector.]

34. A solid occupies a region E with surface S and is immersed in a liquid with constant density ρ. We set up a coordinate system so that the xy-plane coincides with the surface of the liquid, and positive values of z are measured downward into the liquid. Then the pressure at depth z is $p = \rho g z$, where g is the acceleration due to gravity (see Section 8.3). The total buoyant force on the solid due to the pressure distribution is given by the surface integral

$$\mathbf{F} = -\iint_S p\mathbf{n}\, dS$$

where $\mathbf{n}$ is the outer unit normal. Use the result of Exercise 33 to show that $\mathbf{F} = -W\mathbf{k}$, where W is the weight of the liquid displaced by the solid. (Note that $\mathbf{F}$ is directed upward because z is directed downward.) The result is *Archimedes' Principle:* the buoyant force on an object equals the weight of the displaced liquid.

16.10 Summary

The main results of this chapter are all higher-dimensional versions of the Fundamental Theorem of Calculus. To help you remember them, we collect them together here (without hypotheses) so that you can see more easily their essential similarity. Notice that in each case we have an integral of a "derivative" over a region on the left side, and the right side involves the values of the original function only on the *boundary* of the region.

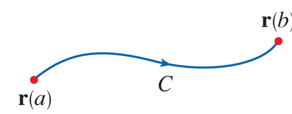
Curves and their boundaries (endpoints)

Fundamental Theorem of Calculus

$$\int_a^b F'(x)\,dx = F(b) - F(a)$$

Fundamental Theorem for Line Integrals

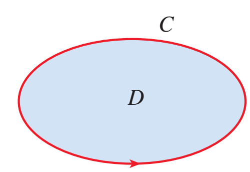
$$\int_C \nabla f \cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a))$$



Surfaces and their boundaries

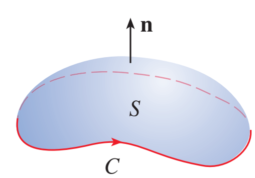
Green's Theorem

$$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA = \int_C P\,dx + Q\,dy$$



Stokes' Theorem

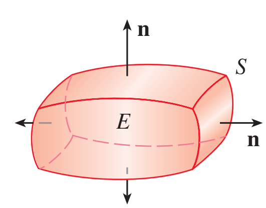
$$\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = \int_C \mathbf{F} \cdot d\mathbf{r}$$



Solids and their boundaries

Divergence Theorem

$$\iiint_E \operatorname{div} \mathbf{F}\,dV = \iint_S \mathbf{F} \cdot d\mathbf{S}$$



16 REVIEW

CONCEPT CHECK

Answers to the Concept Check are available at StewartCalculus.com.

1. What is a vector field? Give three examples that have physical meaning.

2. (a) What is a conservative vector field?
(b) What is a potential function?

3. (a) Write the definition of the line integral of a scalar function f along a smooth curve C with respect to arc length.
(b) How do you evaluate such a line integral?
(c) Write expressions for the mass and center of mass of a thin wire shaped like a curve C if the wire has linear density function $\rho(x, y)$.
(d) Write the definitions of the line integrals along C of a scalar function f with respect to x, y, and z.
(e) How do you evaluate these line integrals?

4. (a) Define the line integral of a vector field $\mathbf{F}$ along a smooth curve C given by a vector function $\mathbf{r}(t)$.
(b) If $\mathbf{F}$ is a force field, what does this line integral represent?
(c) If $\mathbf{F} = \langle P, Q, R\rangle$, what is the connection between the line integral of $\mathbf{F}$ and the line integrals of the component functions P, Q, and R?

5. State the Fundamental Theorem for Line Integrals.

6. (a) What does it mean to say that $\int_C \mathbf{F} \cdot d\mathbf{r}$ is independent of path?
(b) If you know that $\int_C \mathbf{F} \cdot d\mathbf{r}$ is independent of path, what can you say about $\mathbf{F}$?

7. State Green's Theorem.

8. Write expressions for the area enclosed by a curve C in terms of line integrals around C.

9. Suppose $\mathbf{F}$ is a vector field on $\mathbb{R}^3$.
(a) Define curl $\mathbf{F}$. (b) Define div $\mathbf{F}$.
(c) If $\mathbf{F}$ is a velocity field in fluid flow, what are the physical interpretations of curl $\mathbf{F}$ and div $\mathbf{F}$?

10. If $\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j}$, how do you determine whether $\mathbf{F}$ is conservative? What if $\mathbf{F}$ is a vector field on $\mathbb{R}^3$?

11. (a) What is a parametric surface? What are its grid curves?
(b) Write an expression for the area of a parametric surface.
(c) What is the area of a surface given by an equation $z = g(x, y)$?

12. (a) Write the definition of the surface integral of a scalar function f over a surface S.
(b) How do you evaluate such an integral if S is a parametric surface given by a vector function $\mathbf{r}(u, v)$?
(c) What if S is given by an equation $z = g(x, y)$?
(d) If a thin sheet has the shape of a surface S, and the density at (x, y, z) is $\rho(x, y, z)$, write expressions for the mass and center of mass of the sheet.

13. (a) What is an oriented surface? Give an example of a non-orientable surface.
(b) Define the surface integral (or flux) of a vector field $\mathbf{F}$ over an oriented surface S with unit normal vector $\mathbf{n}$.
(c) How do you evaluate such an integral if S is a parametric surface given by a vector function $\mathbf{r}(u, v)$?
(d) What if S is given by an equation $z = g(x, y)$?

14. State Stokes' Theorem.

15. State the Divergence Theorem.

16. In what ways are the Fundamental Theorem for Line Integrals, Green's Theorem, Stokes' Theorem, and the Divergence Theorem similar?

TRUE-FALSE QUIZ

Determine whether the statement is true or false. If it is true, explain why. If it is false, explain why or give an example that disproves the statement.

1. If $\mathbf{F}$ is a vector field, then div $\mathbf{F}$ is a vector field.

2. If $\mathbf{F}$ is a vector field, then curl $\mathbf{F}$ is a vector field.

3. If f has continuous partial derivatives of all orders on $\mathbb{R}^3$, then $\operatorname{div}(\operatorname{curl} \nabla f) = 0$.

4. If f has continuous partial derivatives on $\mathbb{R}^3$ and C is any circle, then $\int_C \nabla f \cdot d\mathbf{r} = 0$.

5. If $\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j}$ and $P_y = Q_x$ in an open region D, then $\mathbf{F}$ is conservative.

6. $\int_{-C} f(x, y)\, ds = -\int_C f(x, y)\, ds$

7. If $\mathbf{F}$ and $\mathbf{G}$ are vector fields and $\operatorname{div}\mathbf{F} = \operatorname{div}\mathbf{G}$, then $\mathbf{F} = \mathbf{G}$.

8. The work done by a conservative force field in moving a particle around a closed path is zero.

9. If $\mathbf{F}$ and $\mathbf{G}$ are vector fields, then

$$\operatorname{curl}(\mathbf{F} + \mathbf{G}) = \operatorname{curl} \mathbf{F} + \operatorname{curl} \mathbf{G}$$

10. If $\mathbf{F}$ and $\mathbf{G}$ are vector fields, then

$$\operatorname{curl}(\mathbf{F}\cdot\mathbf{G}) = \operatorname{curl}\mathbf{F}\cdot\operatorname{curl}\mathbf{G}$$

11. If S is a sphere and $\mathbf{F}$ is a constant vector field, then $\iint_S \mathbf{F}\cdot d\mathbf{S} = 0$.

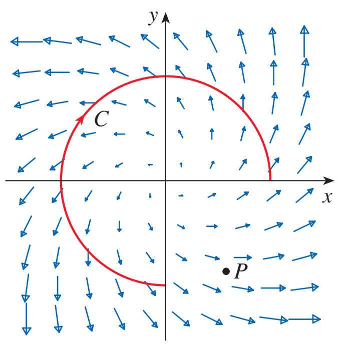
12. There is a vector field $\mathbf{F}$ such that

$$\operatorname{curl}\mathbf{F} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$$

13. The area of the region bounded by the positively oriented, piecewise smooth, simple closed curve C is $A = \oint_C y\,dx$.

EXERCISES

1. A vector field $\mathbf{F}$, a curve C, and a point P are shown.
(a) Is $\int_C \mathbf{F}\cdot d\mathbf{r}$ positive, negative, or zero? Explain.
(b) Is $\operatorname{div}\mathbf{F}(P)$ positive, negative, or zero? Explain.



2–9 Evaluate the line integral.

2. $\int_C x\,ds$,
C is the arc of the parabola $y = x^2$ from $(0, 0)$ to $(1, 1)$

3. $\int_C yz\cos x\,ds$,
C: $x = t,\ y = 3\cos t,\ z = 3\sin t,\ 0 \leqslant t \leqslant \pi$

4. $\int_C y\,dx + (x + y^2)\,dy$, C is the ellipse $4x^2 + 9y^2 = 36$ with counterclockwise orientation

5. $\int_C y^3\,dx + x^2\,dy$, C is the arc of the parabola $x = 1 - y^2$ from $(0, -1)$ to $(0, 1)$

6. $\int_C \sqrt{xy}\,dx + e^y\,dy + xz\,dz$,
C is given by $\mathbf{r}(t) = t^4\,\mathbf{i} + t^2\,\mathbf{j} + t^3\,\mathbf{k}$, $0 \leqslant t \leqslant 1$

7. $\int_C xy\,dx + y^2\,dy + yz\,dz$,
C is the line segment from $(1, 0, -1)$, to $(3, 4, 2)$

8. $\int_C \mathbf{F}\cdot d\mathbf{r}$, where $\mathbf{F}(x, y) = xy\,\mathbf{i} + x^2\,\mathbf{j}$ and C is given by $\mathbf{r}(t) = \sin t\,\mathbf{i} + (1 + t)\,\mathbf{j}$, $0 \leqslant t \leqslant \pi$

9. $\int_C \mathbf{F}\cdot d\mathbf{r}$, where $\mathbf{F}(x, y, z) = e^z\,\mathbf{i} + xz\,\mathbf{j} + (x + y)\,\mathbf{k}$ and C is given by $\mathbf{r}(t) = t^2\,\mathbf{i} + t^3\,\mathbf{j} - t\,\mathbf{k}$, $0 \leqslant t \leqslant 1$

10. Find the work done by the force field

$$\mathbf{F}(x, y, z) = z\,\mathbf{i} + x\,\mathbf{j} + y\,\mathbf{k}$$

in moving a particle from the point $(3, 0, 0)$ to the point $(0, \pi/2, 3)$ along each path.
(a) A straight line
(b) The helix $x = 3\cos t,\ y = t,\ z = 3\sin t$

11–12 Show that $\mathbf{F}$ is a conservative vector field. Then find a function f such that $\mathbf{F} = \nabla f$.

11. $\mathbf{F}(x, y) = (1 + xy)e^{xy}\,\mathbf{i} + (e^y + x^2e^{xy})\,\mathbf{j}$

12. $\mathbf{F}(x, y, z) = \sin y\,\mathbf{i} + x\cos y\,\mathbf{j} - \sin z\,\mathbf{k}$

13–14 Show that $\mathbf{F}$ is conservative and use this fact to evaluate $\int_C \mathbf{F}\cdot d\mathbf{r}$ along the given curve.

13. $\mathbf{F}(x, y) = (4x^3y^2 - 2xy^3)\,\mathbf{i} + (2x^4y - 3x^2y^2 + 4y^3)\,\mathbf{j}$,
C: $\mathbf{r}(t) = (t + \sin\pi t)\,\mathbf{i} + (2t + \cos\pi t)\,\mathbf{j}$, $0 \leqslant t \leqslant 1$

14. $\mathbf{F}(x, y, z) = e^y\,\mathbf{i} + (xe^y + e^z)\,\mathbf{j} + ye^z\,\mathbf{k}$,
C is the line segment from $(0, 2, 0)$ to $(4, 0, 3)$

15. Verify that Green's Theorem is true for the line integral $\int_C xy^2\,dx - x^2y\,dy$, where C consists of the parabola $y = x^2$ from $(-1, 1)$ to $(1, 1)$ and the line segment from $(1, 1)$ to $(-1, 1)$.

16. Use Green's Theorem to evaluate

$$\int_C \sqrt{1 + x^3}\,dx + 2xy\,dy$$

where C is the triangle with vertices $(0, 0)$, $(1, 0)$, and $(1, 3)$.

17. Use Green's Theorem to evaluate $\int_C x^2y\,dx - xy^2\,dy$, where C is the circle $x^2 + y^2 = 4$ with counterclockwise orientation.

18. Find $\operatorname{curl}\mathbf{F}$ and $\operatorname{div}\mathbf{F}$ if

$$\mathbf{F}(x, y, z) = e^{-x}\sin y\,\mathbf{i} + e^{-y}\sin z\,\mathbf{j} + e^{-z}\sin x\,\mathbf{k}$$

19. Show that there is no vector field $\mathbf{G}$ such that

$$\operatorname{curl}\mathbf{G} = 2x\,\mathbf{i} + 3yz\,\mathbf{j} - xz^2\,\mathbf{k}$$

20. If $\mathbf{F}$ and $\mathbf{G}$ are vector fields whose component functions have continuous first partial derivatives, show that

$$\operatorname{curl}(\mathbf{F}\times\mathbf{G}) = \mathbf{F}\operatorname{div}\mathbf{G} - \mathbf{G}\operatorname{div}\mathbf{F} + (\mathbf{G}\cdot\nabla)\mathbf{F} - (\mathbf{F}\cdot\nabla)\mathbf{G}$$

21. If C is any piecewise-smooth simple closed plane curve and f and g are differentiable functions, show that $\int_C f(x)\,dx + g(y)\,dy = 0$.

22. If f and g are twice differentiable functions, show that

$$\nabla^2(fg) = f\nabla^2 g + g\nabla^2 f + 2\nabla f \cdot \nabla g$$

23. If f is a harmonic function, that is, $\nabla^2 f = 0$, show that the line integral $\int f_y\,dx - f_x\,dy$ is independent of path in any simple region D.

24. (a) Sketch the curve C with parametric equations

$$x = \cos t \qquad y = \sin t \qquad z = \sin t \qquad 0 \leqslant t \leqslant 2\pi$$

(b) Find $\int_C 2xe^{2y}\,dx + (2x^2e^{2y} + 2y\cot z)\,dy - y^2\csc^2 z\,dz$.

25. Find the area of the part of the surface $z = x^2 + 2y$ that lies above the triangle with vertices $(0, 0)$, $(1, 0)$, and $(1, 2)$.

26. (a) Find an equation of the tangent plane at the point $(4, -2, 1)$ to the parametric surface S given by

$$\mathbf{r}(u, v) = v^2\,\mathbf{i} - uv\,\mathbf{j} + u^2\,\mathbf{k}$$
$$0 \leqslant u \leqslant 3,\ -3 \leqslant v \leqslant 3$$

(b) Graph the surface S and the tangent plane found in part (a).

(c) Set up, but do not evaluate, an integral for the surface area of S.

(d) If

$$\mathbf{F}(x, y, z) = \frac{z^2}{1 + x^2}\,\mathbf{i} + \frac{x^2}{1 + y^2}\,\mathbf{j} + \frac{y^2}{1 + z^2}\,\mathbf{k}$$

use a computer algebra system to find $\iint_S \mathbf{F} \cdot d\mathbf{S}$ correct to four decimal places.

27–30 Evaluate the surface integral.

27. $\iint_S z\,dS$, where S is the part of the paraboloid $z = x^2 + y^2$ that lies under the plane $z = 4$

28. $\iint_S (x^2z + y^2z)\,dS$, where S is the part of the plane $z = 4 + x + y$ that lies inside the cylinder $x^2 + y^2 = 4$

29. $\iint_S \mathbf{F} \cdot d\mathbf{S}$, where $\mathbf{F}(x, y, z) = xz\,\mathbf{i} - 2y\,\mathbf{j} + 3x\,\mathbf{k}$ and S is the sphere $x^2 + y^2 + z^2 = 4$ with outward orientation

30. $\iint_S \mathbf{F} \cdot d\mathbf{S}$, where $\mathbf{F}(x, y, z) = x^2\,\mathbf{i} + xy\,\mathbf{j} + z\,\mathbf{k}$ and S is the part of the paraboloid $z = x^2 + y^2$ below the plane $z = 1$ with upward orientation

31. Verify that Stokes' Theorem is true for the vector field $\mathbf{F}(x, y, z) = x^2\,\mathbf{i} + y^2\,\mathbf{j} + z^2\,\mathbf{k}$, where S is the part of the paraboloid $z = 1 - x^2 - y^2$ that lies above the xy-plane and S has upward orientation.

32. Use Stokes' Theorem to evaluate $\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S}$, where $\mathbf{F}(x, y, z) = x^2yz\,\mathbf{i} + yz^2\,\mathbf{j} + z^3e^{xy}\,\mathbf{k}$, S is the part of the sphere $x^2 + y^2 + z^2 = 5$ that lies above the plane $z = 1$, and S is oriented upward.

33. Use Stokes' Theorem to evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F}(x, y, z) = xy\,\mathbf{i} + yz\,\mathbf{j} + zx\,\mathbf{k}$ and C is the triangle with vertices $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$, oriented counterclockwise as viewed from above.

34. Use the Divergence Theorem to calculate the surface integral $\iint_S \mathbf{F} \cdot d\mathbf{S}$, where $\mathbf{F}(x, y, z) = x^3\,\mathbf{i} + y^3\,\mathbf{j} + z^3\,\mathbf{k}$ and S is the surface of the solid bounded by the cylinder $x^2 + y^2 = 1$ and the planes $z = 0$ and $z = 2$.

35. Verify that the Divergence Theorem is true for the vector field $\mathbf{F}(x, y, z) = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$, where E is the unit ball $x^2 + y^2 + z^2 \leqslant 1$.

36. Compute the outward flux of

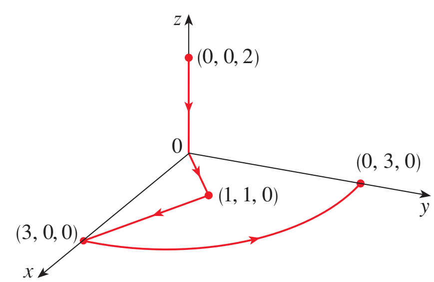
$$\mathbf{F}(x, y, z) = \frac{x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}}{(x^2 + y^2 + z^2)^{3/2}}$$

through the ellipsoid $4x^2 + 9y^2 + 6z^2 = 36$.

37. Let

$$\mathbf{F}(x, y, z) = (3x^2yz - 3y)\,\mathbf{i} + (x^3z - 3x)\,\mathbf{j} + (x^3y + 2z)\,\mathbf{k}$$

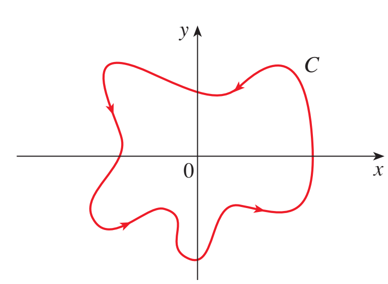
Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$, where C is the curve with initial point $(0, 0, 2)$ and terminal point $(0, 3, 0)$ shown in the figure.



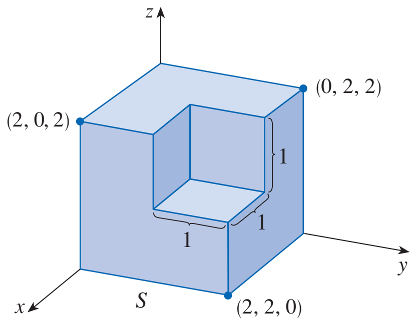
38. Let

$$\mathbf{F}(x, y) = \frac{(2x^3 + 2xy^2 - 2y)\,\mathbf{i} + (2y^3 + 2x^2y + 2x)\,\mathbf{j}}{x^2 + y^2}$$

Evaluate $\oint_C \mathbf{F} \cdot d\mathbf{r}$, where C is shown in the figure.



39. Find $\iint_S \mathbf{F} \cdot \mathbf{n}\, dS$, where $\mathbf{F}(x, y, z) = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$ and S is the outwardly oriented surface shown in the figure (the boundary surface of a cube with a unit corner cube removed).



40. If the components of $\mathbf{F}$ have continuous second partial derivatives and S is the boundary surface of a simple solid region, show that $\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = 0$.

41. If $\mathbf{a}$ is a constant vector, $\mathbf{r} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$, and S is an oriented, smooth surface with a simple, closed, smooth, positively oriented boundary curve C, show that

$$\iint_S 2\mathbf{a} \cdot d\mathbf{S} = \int_C (\mathbf{a} \times \mathbf{r}) \cdot d\mathbf{r}$$