Math 164: Multidimensional Calculus

Midterm 1 ANSWERS

July 13, 2017

Pledge of Honesty

I affirm that I will not give or receive any unauthorized help on this exam, and that all work will be my own.

Signature: ______________________________

Part A

1. (15 points) Consider the vectors $a =< 1, 0, 0 >$, $b =< 1, 0, -1 >$ and $c =< 1, 2, -1 >$.

(a) Compute the scalar projection of b onto c.

Answer:

The scalar projection α of b onto c is given by:

$$\alpha = \frac{b \cdot c}{|c|} = \frac{2}{\sqrt{6}}$$

(b) Find the angle between a and b.

Answer:

We have

$$1 = a \cdot b = |a||b| \cos(\theta) = \sqrt{2} \cos(\theta)$$

so $\theta = (\frac{1}{\sqrt{2}}) = \pi/4$.

(c) A vector $v =< 5, y, z >$ satisfies $a \cdot v = b \cdot v = c \cdot v$. Find y and z.

2. (20 points) Consider the four points $P(0,1,5), Q(1,2,8), R(2,-1,0)$, and $S(1,2,3)$

(a) Find an equation for the plane that passes through points P, Q, and R.

Answer:

•We use the point P and the normal vector $\vec{PQ} \times \vec{PR}$.

•$\vec{PQ} = \langle 1,1,3 \rangle$ and $\vec{PR} = \langle 2,-2,-5 \rangle$

•Then $\vec{PQ} \times \vec{PR} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 3 \\ 2 & -2 & -5 \end{vmatrix} = \begin{vmatrix} 1 & 3 \\ -2 & -5 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 1 & 3 \\ 2 & -5 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 1 & 1 \\ 2 & -2 \end{vmatrix} \mathbf{k} =$ $\langle 1, 11, -4 \rangle$.

•Thus, an equation for the plane is $x+11(y-1)-4(z-5)=0$, i.e. $\boxed{x+11y-4z=-9}$.

•Any multiple of this equation will also work.

(b) Find the area of the triangle with vertices P, Q, and R.

Answer:

•The area of the parallelogram determined by P, Q, and R is $|\vec{PQ} \times \vec{PR}| = |\langle 1, 11, -4 \rangle| = \sqrt{1+121+25} = \sqrt{147}$.

•The area of the triangle is 1/2 the area of the parallelogram, which is $\boxed{\dfrac{\sqrt{147}}{2}}$.

(c) Find the volume of the parallelepiped determined by P, Q, R, and S.

Answer:

•The volume of the parallelepiped is $\vec{PS} \cdot \vec{PQ} \times \vec{PR} = \langle 1,1,-2 \rangle \cdot \langle 1,11,-4 \rangle = 1(1) + 1(11) - 2(-4) = \boxed{20}$.

(d) Find the distance from S to the plane determined by P, Q, and R.

Answer:

•The distance is $D = \dfrac{1(1)+11(2)-4(3)+9}{\sqrt{1+121+25}} = \boxed{\dfrac{20}{\sqrt{147}}}$.

•Alternatively, the distance is the height of the parallelepiped, which is the volume divided by the area of the base, which is $\frac{20}{\sqrt{147}}$.

3. (15 points) Let ℓ_1 be the line that passes through the points $(1, -2, 3)$ and $(2, 0, -1)$, and let ℓ_2 be the line that passes through the point $(3, 1, 2)$ and is perpendicular to the plane $x + 2y + 4z = 0$.

(a) Find symmetric and parametric equations for ℓ_1. Clearly label each set as symmetric or parametric.

Answer:

- We will use the point $P_0(1, -2, 3)$ and the direction $\langle 2-1, 0-(-2), -1-3\rangle = \langle 1, 2, -4\rangle$
- Parametric equations: $\boxed{x = 1 + t,\ y = -2 + 2t,\ z = 3 - 4t}$.
- Symmetric equations: $\boxed{x - 1 = \dfrac{y+2}{2} = \dfrac{z-3}{-4}}$

(b) Find symmetric and parametric equations for ℓ_2. Clearly label each set as symmetric or parametric.

Answer:

- We will use the point $P_0(3, 1, 2)$ and the direction $\mathbf{n} = \langle 1, 2, 4\rangle$
- Parametric equations: $\boxed{x = 3 + t,\ y = -1 + 2t,\ z = 2 + 4t}$.
- Symmetric equations: $\boxed{x - 3 = \dfrac{y-1}{2} = \dfrac{z-2}{4}}$

(c) Are these lines intersecting, parallel, or skew? Briefly explain your answer.

Answer:

- The direction vectors are $\langle 1, 2, -4\rangle$, and $\langle 1, 2, 4\rangle$, which are not parallel so the lines are not parallel
- An intersection point is a solution to

$$1 + t = 3 + s, \quad -2 + 2t = -1 + 2s, \quad 3 - 4t = 2 + 4s$$

From the first equation, we see that any solution requires $t = 2 + s$. From the second equation, we get $t = \frac{1}{2} + s$. This is a contradiction so there is no intersection point. The lines are SKEW.

4. (10 points) Consider the curve $\mathbf{r}(t) = 2t\mathbf{i} + (3-t)\mathbf{j} - 2t\mathbf{k}$.

(a) Find the arc length of the curve between $t = 0$ and $t = 1$.

Answer:

From

$$\mathbf{r}'(t) = 2\mathbf{i} - \mathbf{j} - 2\mathbf{k},$$

we get

$$|\mathbf{r}'(t)| = \sqrt{2^2 + (-1)^2 + (-2)^2} = 3.$$

So,

$$\int_0^1 |\mathbf{r}'(t)|dt = 3t.$$

(b) Reparametrize the curve in terms of arc length s measured from $t = 0$ in the direction of increasing t.

Answer:

$$s = \int_0^t 3dr = 3t.$$

So, $t = \frac{s}{3}$. Substituting t for $\frac{s}{3}$, we get

$$\mathbf{r}(s) = \frac{2}{3}s\mathbf{i} + (3 - \frac{s}{3})\mathbf{j} - \frac{2}{3}s\mathbf{k}.$$

5. (15 points) Consider the curve $\mathbf{r}(t) = \cos\, 2t\mathbf{i} + \sin\, 2t\mathbf{j} + 2t\mathbf{k}$.

(a) Find the unit tangent vector $\mathbf{T}(t)$.

Answer:

We have

$$\mathbf{r}'(t) = -2\sin\, 2t\mathbf{i} + 2\cos\, 2t\mathbf{j} + 2\mathbf{k}$$

and

$$|\mathbf{r}'(t)| = \sqrt{(-2\sin\, 2t)^2 + (2\cos\, 2t)^2 + 2^2} = 2\sqrt{2}.$$

So,

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = -\frac{1}{\sqrt{2}}\sin\, 2t\mathbf{i} + \frac{1}{\sqrt{2}}\cos\, 2t\mathbf{j} + \frac{1}{\sqrt{2}}\mathbf{k}.$$

(b) Find the curvature $\kappa(t)$. Recall the curvature formula $\kappa(t) = \frac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|}$.

Answer:

From

$$\mathbf{T}'(t) = -\frac{2}{\sqrt{2}}\cos\, 2t\mathbf{i} - \frac{2}{\sqrt{2}}\sin\, 2t\mathbf{j},$$

we have

$$|\mathbf{T}'(t)| = \sqrt{2\cos^2\, 2t + 2\sin^2\, 2t} = \sqrt{2}$$

and so

$$\kappa(t) = \frac{1}{2}.$$

6. (15 points) A gun has a muzzle speed of 80 meters per second.

(a) Assuming projectile motion, find the position function $r(t)$. Neglect air resistance and use $g = 9.8m/sec^2$ as the acceleration of gravity.

(b) What angle of elevation should be used to hit an object 200 meters away?

7. (10 points) Evaluate the limit or show that it does not exist.

(a) $\displaystyle \lim_{(x,y)\to(0,0)} \frac{4xy}{x^2+2y^2}$

(b) $\displaystyle \lim_{(x,y)\to(0,0)} \frac{x^3+5y^3}{x^2+y^2}$

Hint: Use polar coordinates: $x = r\cos\theta$, $y = r\sin\theta$.

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