Math 164: Multidimensional Calculus

Midterm 2

June 9, 2022

NAME (please print legibly): ______________________________

Your University ID Number: ______________________________

- You have 75 minutes to work on this exam. You are responsible for checking that this exam has all 11 pages.
- No calculators, phones, electronic devices, books, notes are allowed during the exam.
- **Show all work and justify all answers.** You may not receive full credit for a correct answer if insufficient work is shown or insufficient justification is given.
- Numerical or algebraic simplifications of answers are not required **except when specifically stated otherwise.**
- Please sign the pledge below.
- You must physically write solutions on paper. NO electronic writing on tablets (or similar) is allowed.
- If you have technical issues try not to panic and contact your instructor immediately so we can help you through these issues.
- The zoom session will be recorded. You MUST HAVE A WORKING WEBCAM. It must be on at all times during the exam.
- You may not have any files or windows open on any device during the exam. The instructor may ask you to share your screen at any time during the exam, so be sure to close blackboard and all other files/windows before we start. Your work area must be clear all items/ materials during the exam. The only exception is blank sheets of paper, pens/ pencils.
- When you are done with your exam, please send a private chat message to your proctor before starting your scanning process.

Pledge of Honesty

I affirm that I will not give or receive any unauthorized help on this exam, and that all work will be my own.

Signature: ____________________

QUESTION	VALUE	SCORE
1	10	
2	10	
3	15	
4	15	
5	20	
6	15	
7	15	
TOTAL	100	

Some Formulas

- $$\text{comp}_{\mathbf{u}}(\mathbf{v}) = \frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{u}|} \qquad \text{proj}_{\mathbf{u}}(\mathbf{v}) = \frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{u}|^2}\mathbf{u} \qquad D = \frac{|ax_0+by_0+cz_0+d|}{\sqrt{a^2+b^2+c^2}}$$
- $$\iint_S \mathbf{F}\cdot d\mathbf{S} = \iiint_S \mathbf{F}\cdot\mathbf{n}dS = \iint_D \mathbf{F}\cdot(\mathbf{r_u}\times\mathbf{r_v})dA \qquad \iint_D\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)dA = \int_C Pdx+Qdy$$
- $$\iint_S \text{curl}\mathbf{F}\cdot d\mathbf{S} = \int_C \mathbf{F}\cdot d\mathbf{r} \qquad \iiint_E \text{div}\mathbf{F}dV = \iint_S \mathbf{F}\cdot d\mathbf{S}$$

For spherical coordinates (ρ,θ,ϕ) with $0\le\theta\le 2\pi, 0\le\phi\le\pi$.

- $dV = \rho^2\sin\phi d\rho d\phi d\theta$
- For $\mathbf{r}(\phi,\theta) = a\sin\phi\cos\theta\mathbf{i} + a\sin\phi\sin\theta\mathbf{j} + a\cos\phi\mathbf{k}$:
$$\mathbf{r}_\phi\times\mathbf{r}_\theta = a^2\sin^2\phi\cos\theta\mathbf{i} + a^2\sin^2\phi\sin\theta\mathbf{j} + a^2\sin\phi\cos\phi\mathbf{k}$$

Trig Identities

- $\sin^2\theta+\cos^2\theta=1 \qquad \tan^2+1=\sec^2\theta \qquad \cot^2\theta+1=\csc^2\theta$
- $\sin(2\theta)=2\sin\theta\cos\theta$
- $\sin^2\theta=\frac{1}{2}(1-\cos(2\theta)) \qquad \cos^2(\theta)=\frac{1}{2}(1+\cos(2\theta))$
- $\sin(a+b)=\sin(a)\cos(b)+\cos(a)\sin(b) \qquad \sin(a-b)=\sin(a)\cos(b)-\cos(a)\sin(b)$
- $\cos(a+b)=\cos(a)\cos(b)-\sin(a)\sin(b) \qquad \cos(a-b)=\cos(a)\cos(b)+\sin(a)\sin(b)$
- $\sin(a)\cos(b)=\frac{1}{2}[\sin(a-b)+\sin(a+b)]$
- $\sin(a)\sin(b)=\frac{1}{2}[\cos(a-b)-\cos(a+b)]$
- $\cos(a)\cos(b)=\frac{1}{2}[\cos(a-b)+\cos(a+b)]$

1. (10 points) Let $f(x, y) = x^3 - 3xy + y^2$.

(a) (5 points) Find the directional derivative of f in the direction of the vector $\mathbf{v} = \langle 1, 1\rangle$ at the point $(2, 5)$.

$\nabla f = \langle 3x^2 - 3y, -3x + 2y \rangle$

$\nabla f(2.5) = \langle -3, 4 \rangle$

$\langle -3, 4 \rangle \cdot \frac{1}{\sqrt{2}} \langle 1.1 \rangle = \frac{1}{\sqrt{2}}$

(b) (5 points) Find the unit vector in the direction for which $f(x, y)$ is increasing fastest at the point $(2, 5)$.

$\frac{\langle -3, 4 \rangle}{|\langle -3, 4 \rangle|} = \frac{1}{5} \langle -3, 4 \rangle$

2. (10 points) Find an equation for the tangent plane to the surface $x^3 + y^2 + z = 3$ at the point $(1, 1, 1)$.

Let $f(x, y, z) = x^3 + y^2 + z$

$\nabla f = \langle 3x^2, 2y, 1 \rangle$

$\nabla f(1,1,1) = \langle 3, 2, 1 \rangle$

$3(x-1) + 2(y-1) + (z-1) = 0$

3. (15 points) Consider the function $f(x, y) = 2 - x^4 + 2x^2 - y^2$. Find all the local maximum and minimum values and saddle points of the function.

$\nabla f = \langle -4x^3 + 4x, -2y \rangle$

$-4x^3 + 4x = 0 \quad \rightarrow \quad x(x+1)(x-1) = 0$

$y = 0$

Thus critical points are $(-1, 0)$, $(0, 0)$ and $(1, 0)$

$f_{xx} = -12x^2 + 4 \qquad f_{xy} = f_{yx} = 0 \qquad f_{yy} = -2$

$D(x, y) = (f_{xx})(f_{yy}) - (f_{xy})^2 = 2(12x^2 - 4) = 8(3x^2 - 1)$

$D(1, 0) = 16 > 0$ and $f_{xx}(1, 0) = -8 < 0$

$\rightarrow f(1, 0) = 3$ is a local maximum.

$D(0, 0) = -8 < 0 \rightarrow (0, 0)$ is a saddle point.

$D(-1, 0) = 16 > 0$ and $f_{xx}(-1, 0) = -8 < 0$

$\rightarrow f(-1, 0) = 3$ is a local maximum.

4. (15 points) Use the method of Lagrange multipliers to find the extreme values of the function $f(x, y, z) = x - 2y + 4z$ subject to the given constraint $x^2 + y^2 + 2z^2 \leq 18$.

① $x^2+y^2+2z^2=18.$

Let $g(x,y,z) = x^2+y^2+2z^2$, then

$\nabla f = \langle 1, -2, 4 \rangle$ and $\nabla g = \langle 2x, 2y, 4z \rangle$

$\nabla f = \lambda \nabla g \rightarrow$
$1 = 2\lambda x \rightarrow x = \frac{1}{2\lambda}$
$-2 = 2\lambda y \qquad y = \frac{-1}{\lambda}$
$4 = 4\lambda z \qquad z = \frac{1}{\lambda}$

Since $x^2+y^2+2z^2=18$, $\left(\frac{1}{2\lambda}\right)^2 + \left(\frac{-1}{\lambda}\right)^2 + 2\left(\frac{1}{\lambda}\right)^2 = \frac{1}{\lambda^2} \cdot \frac{13}{4} = 18.$

$\lambda = \pm\sqrt{\frac{13}{72}}$

$x - 2y + 4z = \frac{1}{2\lambda} + \frac{2}{\lambda} + \frac{4}{\lambda} = \frac{13}{2} \cdot \frac{1}{\lambda}$

If $\lambda = +\sqrt{\frac{13}{72}} \rightarrow x - 2y + 4z = \frac{13}{2} \cdot \frac{\sqrt{72}}{\sqrt{13}} = 3\sqrt{26}$

$\lambda = -\sqrt{\frac{13}{72}} \rightarrow x - 2y + 4z = -3\sqrt{26}$

② $x^2+y^2+2z^2 < 18.$

Since $\nabla f = \langle 1, -2, 4 \rangle \neq \langle 0, 0, 0 \rangle$

There is no critical point.

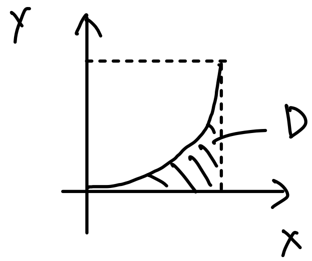
$\therefore$ maximum $3\sqrt{26}$

minimum $-3\sqrt{26}$.

5. (20 points) Set up the double integral for both orders of integration. Then evaluate the double integral using the easier order.

$$\iint_D x(\sqrt{y}+1)dA$$

where D is bounded by $y=0, y=x^2, x=1$.



$D = \{(x,y) : 0 \le x \le 1,\ 0 \le y \le x^2\}$

$$\iint_D x(\sqrt{y}+1)\,dA = \int_0^1 \int_0^{x^2} x(\sqrt{y}+1)\,dy\,dx$$

$D = \{(x,y) : 0 \le y \le 1,\ \sqrt{y} \le x \le 1\}$

$$\iint_D x(\sqrt{y}+1)\,dA = \int_0^1 \int_{\sqrt{y}}^1 x(\sqrt{y}+1)\,dx\,dy$$

$$\int_0^1 \int_0^{x^2} x(\sqrt{y}+1)\,dy\,dx = \int_0^1 \left[x\left(\frac{2}{3}y^{\frac{3}{2}} + y\right)\right]_0^{x^2} dx$$

$$= \int_0^1 \frac{2}{3}x^4 + x^3\,dx$$

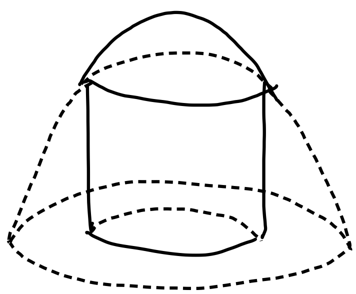
$$= \left[\frac{2}{15}x^5 + \frac{1}{4}x^4\right]_0^1 = \frac{2}{15} + \frac{1}{4} = \frac{23}{60}.$$

6. (15 points) Find the volume of the solid bounded by the plane $z = 0$, the cylinder $x^2 + y^2 = 1$ and the paraboloid $z = 2 - x^2 - y^2$.

In fact, there are two solids bounded by them.

Answer 1) above $z=0$, inside of $x^2+y^2=1$, below $z=2-x^2-y^2$

$$V = \iint_D 2-x^2-y^2\, dA \quad \text{where } D=\{(x,y): x^2+y^2 \le 1\}$$



In polar coordinates,

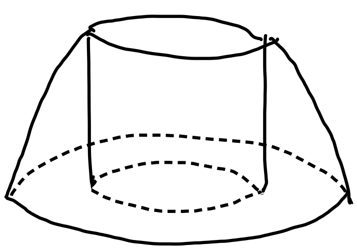
$$D=\{(r,\theta) : 0 \le r \le 1,\ 0 \le \theta \le 2\pi\}$$

$$2-x^2-y^2 = 2-r^2$$

$$V = \int_0^1 \int_0^{2\pi} (2-r^2)\cdot r\, d\theta\, dr = \left(\int_0^{2\pi} 1\, d\theta\right)\left(\int_0^1 2r - r^3\, dr\right)$$

$$= 2\pi \left[r^2 - \tfrac{1}{4}r^4\right]_0^1 = 2\pi \cdot \tfrac{3}{4} = \tfrac{3}{2}\pi$$

Answer 2) above $z=0$, outside of $x^2+y^2=1$, below $z=2-x^2-y^2$



$$D=\{(r,\theta) : 1 \le r \le \sqrt{2},\ 0 \le \theta \le 2\pi\}$$

$$V = \iint_D 2-x^2-y^2\, dA = \int_1^{\sqrt{2}} \int_0^{2\pi} (2-r^2)\, r\, d\theta\, dr$$

$$= 2\pi \int_1^{\sqrt{2}} 2r - r^3\, dr = 2\pi \left[r^2 - \tfrac{1}{4}r^4\right]_1^{\sqrt{2}}$$

$$= \frac{\pi}{2}.$$

Both of them are answers.

7. (15 points) Evaluate the triple integral

$$\iiint_E (x^2 + y^2)^{1/2} dV$$

where E lies between the spheres $x^2 + y^2 + z^2 = 4$ and $x^2 + y^2 + z^2 = 9$.

When we use spherical coordinates,

$$E = \{ (r, \theta, \phi) : 2 \le r \le 3,\ 0 \le \theta \le 2\pi,\ 0 \le \phi \le \pi \}$$

$$(x^2+y^2)^{\frac{1}{2}} = \left[(\rho \cos\theta \sin\phi)^2 + (\rho \sin\theta \sin\phi)^2 \right]^{\frac{1}{2}}$$

$$= \rho \sin\phi$$

$$\iiint_E (x^2+y^2)^{\frac{1}{2}} dV = \int_0^{\pi} \int_0^{2\pi} \int_2^3 \rho \sin\phi \cdot \rho^2 \sin\phi \, d\rho \, d\theta \, d\phi$$

$$= \int_0^{\pi} \int_0^{2\pi} \int_2^3 \rho^3 \sin^2\phi \, d\rho \, d\theta \, d\phi$$

$$= \left(\int_0^{\pi} \sin^2\phi \, d\phi \right) \left(\int_0^{2\pi} 1 \, d\theta \right) \left(\int_2^3 \rho^3 \, d\rho \right)$$

$$\int_0^{\pi} \sin^2\phi \, d\phi = \int_0^{\pi} \frac{1}{2} (1 - \cos(2\phi)) \, d\phi = \left[\frac{1}{2}\phi - \frac{1}{4} \sin(2\phi) \right]_0^{\pi} = \frac{\pi}{2}.$$

$$\int_0^{2\pi} 1 \, d\theta = 2\pi$$

$$\int_2^3 \rho^3 \, d\rho = \left[\frac{1}{4} \rho^4 \right]_2^3 = \frac{1}{4} (81 - 16) = \frac{65}{4}$$

$$\therefore \iiint_E (x^2+y^2)^{\frac{1}{2}} dV = \frac{\pi}{2} \cdot 2\pi \cdot \frac{65}{4} = \frac{65}{4} \pi^2$$

Blank page for scratch work