Math 164: Multidimensional Calculus

Midterm 1

May 26, 2022

NAME (please print legibly): ______________________________

Your University ID Number: ______________________________

- You have 75 minutes to work on this exam. You are responsible for checking that this exam has all 11 pages.
- No calculators, phones, electronic devices, books, notes are allowed during the exam.
- **Show all work and justify all answers.** You may not receive full credit for a correct answer if insufficient work is shown or insufficient justification is given.
- Numerical or algebraic simplifications of answers are not required **except when specifically stated otherwise.**
- Please sign the pledge below.
- You must physically write solutions on paper. NO electronic writing on tablets (or similar) is allowed.
- If you have technical issues try not to panic and contact your instructor immediately so we can help you through these issues.
- The zoom session will be recorded. You MUST HAVE A WORKING WEBCAM. It must be on at all times during the exam.
- You may not have any files or windows open on any device during the exam. The instructor may ask you to share your screen at any time during the exam, so be sure to close blackboard and all other files/windows before we start. Your work area must be clear all items/ materials during the exam. The only exception is blank sheets of paper, pens/ pencils.
- When you are done with your exam, please send a private chat message to your proctor before starting your scanning process.

Pledge of Honesty

I affirm that I will not give or receive any unauthorized help on this exam, and that all work will be my own.

Signature: ______________________________

QUESTION	VALUE	SCORE
1	15	
2	10	
3	15	
4	15	
5	15	
6	15	
7	15	
TOTAL	100	

Some Formulas

- $$\text{comp}_{\mathbf{u}}(\mathbf{v}) = \frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{u}|} \qquad \text{proj}_{\mathbf{u}}(\mathbf{v}) = \frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{u}|^2}\mathbf{u} \qquad D = \frac{|ax_0+by_0+cz_0+d|}{\sqrt{a^2+b^2+c^2}}$$
- $$\iint_S \mathbf{F}\cdot d\mathbf{S} = \iiint_S \mathbf{F}\cdot\mathbf{n}\,dS = \iint_D \mathbf{F}\cdot(\mathbf{r_u}\times\mathbf{r_v})\,dA \qquad \iint_D\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)dA = \int_C P\,dx+Q\,dy$$
- $$\iint_S \text{curl}\mathbf{F}\cdot d\mathbf{S} = \int_C \mathbf{F}\cdot d\mathbf{r} \qquad \iiint_E \text{div}\mathbf{F}\,dV = \iint_S \mathbf{F}\cdot d\mathbf{S}$$

For spherical coordinates (ρ,θ,ϕ) with $0\le\theta\le 2\pi, 0\le\phi\le\pi$.

- $dV = \rho^2\sin\phi\,d\rho\,d\phi\,d\theta$
- For $\mathbf{r}(\phi,\theta) = a\sin\phi\cos\theta\mathbf{i} + a\sin\phi\sin\theta\mathbf{j} + a\cos\phi\mathbf{k}$:

$$\mathbf{r}_\phi\times\mathbf{r}_\theta = a^2\sin^2\phi\cos\theta\mathbf{i} + a^2\sin^2\phi\sin\theta\mathbf{j} + a^2\sin\phi\cos\phi\mathbf{k}$$

Trig Identities

- $\sin^2\theta+\cos^2\theta = 1 \qquad \tan^2+1 = \sec^2\theta \qquad \cot^2\theta+1 = \csc^2\theta$
- $\sin(2\theta) = 2\sin\theta\cos\theta$
- $\sin^2\theta = \frac{1}{2}(1-\cos(2\theta)) \qquad \cos^2(\theta) = \frac{1}{2}(1+\cos(2\theta))$
- $\sin(a+b) = \sin(a)\cos(b)+\cos(a)\sin(b) \qquad \sin(a-b) = \sin(a)\cos(b)-\cos(a)\sin(b)$
- $\cos(a+b) = \cos(a)\cos(b)-\sin(a)\sin(b) \qquad \cos(a-b) = \cos(a)\cos(b)+\sin(a)\sin(b)$
- $\sin(a)\cos(b) = \frac{1}{2}[\sin(a-b)+\sin(a+b)]$
- $\sin(a)\sin(b) = \frac{1}{2}[\cos(a-b)-\cos(a+b)]$
- $\cos(a)\cos(b) = \frac{1}{2}[\cos(a-b)+\cos(a+b)]$

1. (15 points) Let $\mathbf{a} = \langle 1, 2, 3\rangle$, $\mathbf{b} = \langle -1, 1, 2\rangle$ and $\mathbf{c} = \langle 2, 1, 4\rangle$. (Note that they are vectors not points.)

(a) (5 points) Find the vector projection of $\mathbf{a}$ onto $\mathbf{b}$.

$$\text{proj}_b a = \frac{a \cdot b}{|b|^2} b$$

$$a \cdot b = \langle 1, 2, 3\rangle \cdot \langle -1, 1, 2\rangle = -1 + 2 + 6 = 7$$

$$|b|^2 = 1 + 1 + 2^2 = 6$$

$$\text{proj}_b a = \frac{7}{6} \langle -1, 1, 2\rangle$$

(b) (5 points) Find the area of the triangle determined by vectors $\mathbf{a}$ and $\mathbf{b}$. (This is the triangle with vertices $(0, 0, 0)$, $(1, 2, 3)$ and $(-1, 1, 2)$.)

$$a \times b = \begin{vmatrix} i & j & k \\ 1 & 2 & 3 \\ -1 & 1 & 2 \end{vmatrix} = i \begin{vmatrix} 2 & 3 \\ 1 & 2 \end{vmatrix} - j \begin{vmatrix} 1 & 3 \\ -1 & 2 \end{vmatrix} + k \begin{vmatrix} 1 & 2 \\ -1 & 1 \end{vmatrix} = i - 5j + 3k$$

the area of the triangle $= \frac{1}{2} |a \times b| = \frac{1}{2}\sqrt{1 + 25 + 9} = \frac{1}{2}\sqrt{35}$

(c) (5 points) Find the volume of the parallelepiped determined by the vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$.

the volume $= |(a \times b) \cdot c| = |\langle 1, -5, 3\rangle \cdot \langle 2, 1, 4\rangle|$

$$= |2 - 5 + 12| = 9$$

2. (10 points) Let L be the line that passes through the points $(3, -2, 1)$ and $(-1, 0, 2)$.

(a) (5 points) Find symmetric and parametric equations for the line L.

$V = \langle 3, -2, 1 \rangle - \langle -1, 0, 2 \rangle = \langle 4, -2, -1 \rangle$

$r_0 = \langle -1, 0, 2 \rangle$

parametric equations $x = 4t - 1$

$y = -2t$

$z = -t + 2$

Symmetric equations $\dfrac{x+1}{4} = \dfrac{y}{-2} = \dfrac{z-2}{-1}$

(b) (5 points) At what point does this line L intersect the xy-plane?

$z = -t + 2 = 0 \rightarrow t = 2$

Since $x = 4t - 1$, $y = -2t$

$(x, y, z) = (7, -4, 0)$

3. (15 points) Let P be the plane that passes through the points $(2,1,2)$, $(1,1,0)$ and $(0,0,1)$.

(a) (10 points) Find an equation of the plane P.

$\langle 2.1.2\rangle - \langle 1.1.0\rangle = \langle 1,0,2\rangle$

$\langle 1.1.0\rangle - \langle 0.0.1\rangle = \langle 2.1.1\rangle$

$$\langle 1.0,2\rangle \times \langle 2 1.1\rangle = \begin{vmatrix} i & j & k \\ 1 & 0 & 2 \\ 2 & 1 & 1 \end{vmatrix} = i\begin{vmatrix} 0 & 2 \\ 1 & 1 \end{vmatrix} - j\begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix} + k\begin{vmatrix} 1 & 0 \\ 2 & 1 \end{vmatrix}$$

$$= -2i + 3j + k$$

$n = \langle -2,3,1\rangle$

The equation of the plane is

$$-2x + 3y + (z-1) = 0$$

(b) (5 points) Find the distance from the origin to the plane P.

Distance from $(0.0.0)$ to $-2x+3y+z-1=0$

$$D = \frac{|-2\cdot 0 + 3\cdot 0 + 0 - 1|}{\sqrt{2^2+3^2+1^2}} = \frac{1}{\sqrt{14}}$$

4. (15 points) Suppose the curve C is parametrized by $\mathbf{r}(t) = \langle \cos(t), \sin(t), t \rangle$.

(a) (8 points) Find parametric equations for the tangent line to the curve C at the point $(-1, 0, \pi)$.

$r(t) = \langle \cos(t), \sin(t), t \rangle = \langle -1, 0, \pi \rangle$ when $t = \pi$

$r'(t) = \langle -\sin t, \cos t, 1 \rangle$

$r'(\pi) = \langle 0, -1, 1 \rangle \rightarrow$ the direction of the line

$$\begin{aligned} x &= 0 \cdot t - 1 \\ y &= -1 \cdot t + 0 \\ z &= 1 \cdot t + \pi \end{aligned} \quad \rightarrow \quad \begin{aligned} x &= -1 \\ y &= -t \\ z &= t + \pi \end{aligned}$$

(b) (7 points) Find the length of the arc of the curve C from the point $(1, 0, 0)$ to the point $(-1, 0, \pi)$.

t is from 0 to π

$$\int_0^{\pi} |r'(t)|\, dt = \int_0^{\pi} \sqrt{(-\sin t)^2 + (\cos(t))^2 + 1}\, dt$$

$$= \int_0^{\pi} \sqrt{2}\, dt = \sqrt{2}\pi$$

5. (15 points) Answer each part below, showing all work.

(a) (5 points) Evaluate the limit or give a clear argument that it does not exist.

$$\lim_{(x,y)\to(0,0)} (x-1)^2y^2$$

$(x-1)^2y^2$ is a polynomial. so it is continuous.

$$\lim_{(x,y)\to(0,0)} (x-1)^2y^2 = (0-1)^2\cdot 0^2 = 0$$

(b) (5 points) Evaluate the limit or give a clear argument that it does not exist.

$$\lim_{(x,y)\to(0,0)} \frac{(x-y)^2}{x^2+y^2}$$

1) $y=0$ $\quad \lim_{x\to 0} \frac{x^2}{x^2} = 1$ $\quad 1 \neq 0$

2) $x=y$ $\quad \lim_{x\to 0} \frac{0}{2x^2} = 0$ $\quad$ The limit does not exist.

(c) (5 points) Evaluate the limit or give a clear argument that it does not exist.

$$\lim_{(x,y)\to(0,0)} \frac{x^2y^2(x-y)^2}{x^2+y^2}$$

Solution 1) $\left|\frac{x^2y^2(x-y)^2}{x^2+y^2}\right| \leq \left|\frac{x^2y^2(x-y)^2}{x^2}\right| = |y^2(x-y)^2|$

$\lim_{(x,y)\to(0,0)} |y^2(x-y)^2| = 0$ since $y^2(x-y)^2$ is continuous.

By squeeze theorem the limit is 0.

Solution 2) $x = r\cos\theta$ $\quad y = r\sin\theta$

$$\lim_{(x,y)\to(0,0)} \frac{x^2y^2(x-y)^2}{x^2+y^2} = \lim_{r\to 0} r^4\cos^2\theta\sin^2\theta(\cos\theta-\sin\theta)^2 = 0$$

The limit is 0.

6. (15 points) Let $f(x, y, z) = x^3 + xe^{yz}$. Compute the following partial derivatives.

(a) (5 points) $f_z =$ $x e^{yz} \cdot \frac{d}{dz}(yz) = x y e^{yz}$

(b) (5 points) $f_{xy} =$

$f_x = 3x^2 + e^{yz}$

$f_{xy} = z e^{yz}$

(c) (5 points) $f_{yx} =$

$f_y = x z e^{yz}$

$f_{yx} = z e^{yz}$

7. (15 points) Let $f(x, y) = x^3 + y^3$.

(a) (10 points) Find an equation of the tangent plane to the surface $z = f(x, y)$ at the point $P(2, 1, 9)$.

$f_x = 3x^2 \qquad f_y = 3y^2 \quad \rightarrow f_x(2.1) = 12 \qquad f_y(2.1) = 3$

The tangent plane is

$z - 9 = 12(x-2) + 3(y-1)$

(b) (5 points) Use the result of part (a) to approximate the value of $f(2.01, 0.99)$.

The linearization of $f(x\ y)$ at (2.1) is

$L(x.y) = 9 + 12(x-2) + 3(y-1)$

$L(2.01.\ 0.99) = 9 + 12(0.01) + 3(-0.01)$

$= 9.09$

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