

# Math 164: Multidimensional Calculus

Midterm II

November 17, 2016

NAME (please print legibly): \_\_\_\_\_

Your University ID Number: \_\_\_\_\_

*SOLUTIONS*

Indicate your instructor with a check in the appropriate box:

Kleene	TR 12:30-1:45pm	<input type="checkbox"/>
Salur	MW 3:25-4:40pm	<input type="checkbox"/>
Gafni	TR 3:25-4:40pm	<input type="checkbox"/>
Lee	MWF 09:00-09:50am	<input type="checkbox"/>

- You have 75 minutes to work on this exam.
- You are responsible for checking that this exam has all 10 pages.
- No calculators, phones, electronic devices, books, notes are allowed during the exam.
- Show all work and justify all answers.
- Please sign the pledge below.

## Pledge of Honesty

I affirm that I will not give or receive any unauthorized help on this exam, and that all work will be my own.

Signature: \_\_\_\_\_

QUESTION	VALUE	SCORE
1	10	
2	15	
3	10	
4	15	
5	25	
6	15	
7	10	
TOTAL	100	

1. (10 points) Consider the function

$$f(x, y, z) = \frac{2x^2}{x+z} + x^2y^2e^z - xy\sin(z) + \frac{e^{\sqrt{x+y}}}{\cos(xy)}$$

don't want to differentiate that!  
Use Clairaut!

Evaluate the third order partial derivative  $f_{xyz}(1, 1, 0)$ .

$$f_z = -2x^2(x+z)^{-2} + x^2y^2e^z - xy\cos(z) + 0$$

$$\Rightarrow (f_z)_y = 0 + 2x^2ye^z - x\cos(z)$$

$$\Rightarrow f_{zyx} = 4xye^z - \cos(z)$$

$$\begin{aligned} \Rightarrow f_{zyx}(1, 1, 0) &= 4 \cdot 1 \cdot 1 \cdot e^0 - \cos(0) \\ &= 4 - 1 \\ &= \boxed{3} \end{aligned}$$

We can see all mixed partials will be continuous at  $(1, 1, 0)$  since all functions involved are continuous everywhere except where their denominators = 0, which could at worst be where  $x+z=0$  or  $\cos(xy)=0$ , neither of which is true for  $(1, 1, 0)$ . Thus, by Clairaut, all mixed partials are equal at  $(1, 1, 0)$ , so  $f_{xyz}(1, 1, 0) = f_{zyx}(1, 1, 0) = \boxed{3}$

2. (15 points) Let  $f(x, y)$  be given by

$$f(x, y) = 4 - \ln(xy - 9) + x^2 \cos(\pi y).$$

a) Explain why  $f(x, y)$  is differentiable at  $(5, 2)$ .

$$f_x = -\frac{y}{xy-9} + 2x \cos(\pi y)$$

$$f_y = -\frac{x}{xy-9} - \pi x^2 \sin(\pi y)$$

If the partial derivs exist <sup>near  $(a,b)$</sup>  and are continuous ~~where~~ at  $(a,b)$  then  $f$  is differentiable at  $(a,b)$ . In this case,  $f_x$  and  $f_y$  are continuous away from  $xy=9$ , so  $f$  exist and is differentiable at  $(5,2)$  since  $5 \cdot 2 \neq 9$ .

b) Find an equation for the linearization of  $f$  at the point  $(5, 2)$ , and use it to estimate the value of  $f$  at the point  $(5.05, 1.98)$ .

$$\begin{aligned} L(x,y) &= f_x(5,2)(x-5) + f_y(5,2)(y-2) + f(5,2) \\ &= \left[ -\frac{2}{10-9} + 10(-1) \right] (x-5) + \frac{-5}{10-9} (y-2) + (4 - \ln(1) + 25(-1)) \\ &= -12(x-5) - 5(y-2) + (-21) \end{aligned}$$

Near  $(5,2)$ ,  $f(x,y) \approx L(x,y)$  so

$$\begin{aligned} f(5.05, 1.98) &\approx L(5.05, 1.98) = -12(.05) - 5(-.02) - 21 \\ &= -.60 - .10 - 21 \end{aligned}$$

$$f(5.05, 1.98) \approx -21.7$$

$$= \boxed{-21.7}$$

3. (10 points) Write an equation for the tangent plane to the surface  $x^2 + 2y^2 + 3z^2 = 6$  at the point  $(1, 1, 1)$ .

$$f(x, y, z) = x^2 + 2y^2 + 3z^2$$

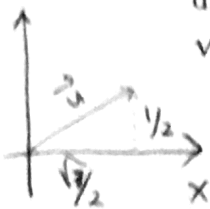
$$\nabla f = \langle f_x, f_y, f_z \rangle = \langle 2x, 4y, 6z \rangle$$

$$\nabla f(1, 1, 1) = \langle 2, 4, 6 \rangle = \vec{n} = \text{normal vector to tangent plane at } (1, 1, 1)$$

$$\vec{n} \cdot \langle x-1, y-1, z-1 \rangle = 0$$

$$2(x-1) + 4(y-1) + 6(z-1) = 0$$

4. (15 points) Find the directional derivative of  $f(x, y) = 2x^2y^3 + 6xy$  at  $(1, 1)$  in the direction of a unit vector whose angle with the positive  $x$ -axis is  $\pi/6$ . (Recall that  $\cos(\pi/6) = \sqrt{3}/2$ )



$$\vec{u} = \left\langle \frac{\sqrt{3}}{2}, \frac{1}{2} \right\rangle$$

$$\begin{aligned} D_{\vec{u}} f(1, 1) &= \nabla f(1, 1) \cdot \vec{u} \\ &= f_x(1, 1) \cos \theta + f_y(1, 1) \sin \theta \\ &= 10 \cdot \frac{\sqrt{3}}{2} + 12 \cdot \frac{1}{2} \\ &= \boxed{5\sqrt{3} + 6} \end{aligned}$$

$$\left. \begin{aligned} f_x &= 4xy^3 + 6y \rightarrow f_x(1, 1) = 4 + 6 = 10 \\ f_y &= 6x^2y^2 + 6x \rightarrow f_y(1, 1) = 6 + 6 = 12 \end{aligned} \right\} \Rightarrow \nabla f = \langle 10, 12 \rangle \text{ at } (1, 1)$$

5. (25 points) a) Consider the function  $f(x, y) = x^2y - 2xy + 2y^2 - 15y$ . Find all the critical points of  $f$  and classify them as relative maxima, relative minima, or saddle points.

$$f_x = 2xy - 2y + 0 - 0 = 2xy - 2y = 2(x-1)y$$

$$\Rightarrow f_x = 0 \text{ when } x=1 \text{ or } y=0$$

$$f_y = x^2 - 2x + 4y - 15$$

$$\Rightarrow f_y = 0 \text{ when } x^2 - 2x + 4y - 15 = 0$$

$$\text{If } \underline{x=1}, f_y = 0 \Rightarrow 4y - 16 = 0 \Rightarrow y = 4 \quad \boxed{(1, 4)}$$

$$\text{If } \underline{y=0}, f_y = 0 \Rightarrow x^2 - 2x - 15 = 0 \Rightarrow (x+3)(x-5) = 0$$

$$\Rightarrow x = -3 \text{ or } x = 5$$

$$\boxed{(-3, 0)} \text{ and } \boxed{(5, 0)}$$

$f_x$  and  $f_y$  are defined everywhere so the only critical pts. are where  $f_x = 0 = f_y$ :  $(1, 4)$ ,  $(-3, 0)$ , and  $(5, 0)$

$$f_{xx} = 2y; \quad f_{yy} = 4; \quad f_{xy} = f_{yx} = 2x - 2 \Rightarrow D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = f_{xx}f_{yy} - f_{xy}^2$$

$$\text{At } (1, 4), D = 32 - 0 = 32 > 0 \text{ and } f_{xx} = 8 > 0$$

so  $\boxed{(1, 4)}$  is a rel. min.

$$= 8y - (2x-2)^2 \\ = 8y - (4x^2 - 8x + 4)$$

$$\text{At } (-3, 0), D = 0 - (2(-3) - 2)^2 < 0, \text{ so } \boxed{(-3, 0)} \text{ is a saddle pt.}$$

$$\text{At } (5, 0), D = 0 - (2(5) - 2)^2 < 0, \text{ so } \boxed{(5, 0)} \text{ is a saddle pt.}$$

b) Find the extreme values of the function  $f(x, y) = 3x^2 + 2y^2 - 4y$  subject to the constraint  $x^2 + y^2 \leq 9$ . Let  $D = \{(x, y) \mid x^2 + y^2 \leq 9\}$ .  $D$  is closed and bdd so extrema are attained.

On the boundary, use Lagrange multipliers ( $g(x, y) = x^2 + y^2 = 9$ ):

$$\begin{cases} \nabla f = \langle 6x, 4y - 4 \rangle \\ \nabla g = \langle 2x, 2y \rangle \end{cases} \Rightarrow \nabla f = \lambda \nabla g \text{ gives } \begin{cases} 6x = \lambda 2x \\ 4y - 4 = \lambda 2y \end{cases}$$

$$\textcircled{1} \quad 6x = \lambda 2x \Rightarrow 2x(3 - \lambda) = 0 \Rightarrow x = 0 \text{ or } \lambda = 3$$

$$\textcircled{2} \quad 4y - 4 = \lambda 2y \Rightarrow 4y - 2\lambda y - 4 = 0$$

$$\textcircled{3} \quad x^2 + y^2 = 9$$

$$\textcircled{1} \text{ and } \textcircled{2} \Rightarrow \text{if } \lambda = 3, 4y - 6y - 4 = 0 \Rightarrow -2y = 4 \Rightarrow y = -2 \xrightarrow{\text{by } \textcircled{3}} x = \pm \sqrt{5} \Rightarrow$$

$$\begin{array}{l} \boxed{(\sqrt{5}, -2)} \rightarrow f = 15 - 4 + 8 = \boxed{19} \\ \boxed{(-\sqrt{5}, -2)} \rightarrow f = \boxed{19} \end{array}$$

$$\textcircled{1} \text{ and } \textcircled{3} \Rightarrow \text{if } x = 0, y^2 = 9 \Rightarrow y = \pm 3 \Rightarrow$$

$$\begin{array}{l} \boxed{(0, 3)} \rightarrow f = 0 + 18 - 12 = \boxed{6} \\ \boxed{(0, -3)} \rightarrow f = 0 + 18 + 12 = \boxed{30} \end{array}$$

On the interior, check for critical points:  $f_x = 0$  when  $x = 0$   
 $f_y = 0$  when  $4y = 4, \text{ or } y = 1$

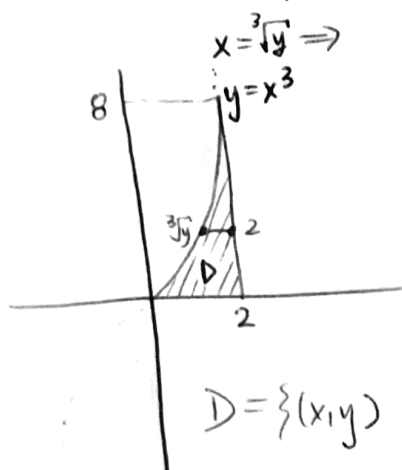
$$\boxed{(0, 1)} \rightarrow f = 0 + 2 - 4 = \boxed{-2}$$

So, we must compare the values  $-2, 6, 19,$  and  $30$

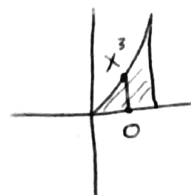
The abs. max. of  $f$  on  $D$  is  $30$ , attained at  $(0, -3)$ .

The abs. min. of  $f$  on  $D$  is  $-2$ , attained at  $(0, 1)$ .

6. (15 points) Evaluate the integral by reversing the order of integration.



$$I = \int_0^8 \int_{\sqrt[3]{y}}^2 e^{x^4} dx dy = \int_0^2 \int_0^{x^3} e^{x^4} dy dx$$



$$D = \{(x, y) \mid 0 \leq y \leq 8, \sqrt[3]{y} \leq x \leq 2\}$$

$$D = \{(x, y) \mid 0 \leq 2 \leq x, 0 \leq y \leq x^3\}$$

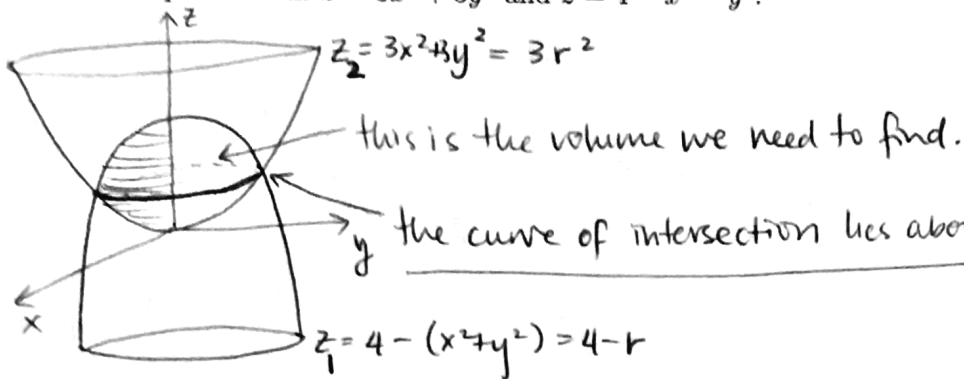
$$I = \int_0^2 \left[ y e^{x^4} \right]_0^{x^3} dx = \int_0^2 x^3 e^{x^4} dx = \left[ \frac{1}{4} e^{x^4} \right]_0^2 = \frac{1}{4} (e^{16} - e^0)$$

$\uparrow$   
 $u = x^4$   
 $\frac{1}{4} du = x^3 dx$

$$= \boxed{\frac{1}{4} (e^{16} - 1)}$$



7. (10 points) Use polar coordinates to find the volume of the solid bounded by the paraboloids  $z = 3x^2 + 3y^2$  and  $z = 4 - x^2 - y^2$ .



$$3x^2 + 3y^2 = 4 - x^2 - y^2$$

$$\text{or } 4x^2 + 4y^2 = 4$$

$$\text{or } x^2 + y^2 = 1$$

$$\text{or } r = 1$$

$$\text{so } D = \{(x, y) \mid x^2 + y^2 \leq 1\}$$

$$\text{or } D = \{(r, \theta) \mid 0 \leq r \leq 1, 0 \leq \theta \leq 2\pi\}$$

$$\text{Volume} = \iint_D (z_1 - z_2) dA$$

$$= \int_0^{2\pi} \int_0^1 ((4-r) - 3r^2) r dr d\theta$$

$$= \int_0^{2\pi} \left( 4r - r^2 - 3r^3 \right) dr d\theta = \int_0^{2\pi} \left[ 2r^2 - \frac{1}{3}r^3 - \frac{3}{4}r^4 \right]_0^1 d\theta$$

$$= \left( 2 - \frac{1}{3} - \frac{3}{4} \right) 2\pi = \boxed{\frac{26}{12} \pi}$$