Math 164: Multidimensional Calculus

Midterm 2

November 17, 2015

NAME (please print legibly): SOLUTIONS
Your University ID Number: ______________________
Indicate your instructor with a check in the appropriate box:

Bobkova	TR 12:30-1:45pm	
Chen	MW 3:25-4:40pm	
Dummit	TR 3:25-4:40pm	
Salur	MWF 09:00-09:50am	

- You are responsible for checking that this exam has all 9 pages.
- No calculators, phones, electronic devices, books, notes are allowed during the exam.
- Show all work and justify all answers.
- Please sign the pledge below.

Pledge of Honesty

I affirm that I will not give or receive any unauthorized help on this exam, and that all work will be my own.

Signature: ______________________

QUESTION	VALUE	SCORE
1	8	
2	8	
3	12	
4	15	
5	14	
6	14	
7	14	
8	15	
TOTAL	100	

1. (8 points) Suppose the equation $x^2z^4 + 2ye^{x+z} = 5$ defines z implicitly as a function of x and y. Find the value of $\frac{\partial z}{\partial x}$ at the point $(x, y, z) = (-1, 2, 1)$.

Let $F(x,y,z) = x^2z^4 + 2ye^{x+z} - 5$. Then $\frac{\partial z}{\partial x} = -\frac{F_x}{F_z}$.

$F_x = 2xz^4 + 2ye^{x+z} \Rightarrow F_x(-1,2,1) = -2 + 4e^0 = 2$

$F_z = 4x^2z^3 + 2ye^{x+z} \Rightarrow F_z(-1,2,1) = 4 + 4e^0 = 8$

So $\frac{\partial z}{\partial x} = \frac{-2}{8} = \boxed{-\frac{1}{4}}$ at $(-1,2,1)$

2. (8 points) Find an equation for the tangent plane to the surface $xy^2z + \ln(x + 2y + z) = 2$ at the point $(x, y, z) = (2, -1, 1)$.

$F(x,y,z) = xy^2z + \ln(x+2y+z) \Rightarrow \nabla F = \langle y^2z + \frac{1}{x+2y+z}, 2xyz + \frac{2}{x+2y+z}, xy^2 + \frac{1}{x+2y+z} \rangle$

$\Rightarrow \nabla F(2,-1,1) = \langle 1 + \frac{1}{1}, -4 + \frac{2}{1}, 2 + \frac{1}{1} \rangle = \langle 2, -2, 3 \rangle = \vec{n}$

$\vec{n} \cdot \langle x-2, y+1, z-1 \rangle = 2(x-2) + (-2)(y+1) + 3(z-1) = 0$

or $\boxed{2x - 2y + 3z = 9}$

3. (12 points) Consider the function $f(x, y) = xy^2$ and the point $P(1, 2)$.

(a) Find the rate of change of f at P in the direction of the vector $\mathbf{u} = \left\langle \frac{2}{\sqrt{13}}, \frac{3}{\sqrt{13}} \right\rangle$.

$\nabla f = \langle y^2, 2xy \rangle \Rightarrow \nabla f(1,2) = \langle 4, 4 \rangle$

$\Rightarrow D_{\hat{u}} f = \nabla f \cdot \vec{u} = \langle 4, 4 \rangle \cdot \frac{1}{\sqrt{13}} \langle 2, 3 \rangle = \frac{1}{\sqrt{13}} (8 + 12) = \frac{20}{\sqrt{13}}$

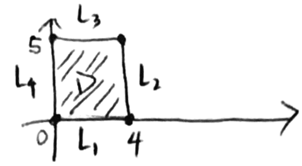
(b) Find the maximal rate of change of f at P and the direction in which it occurs.

max rate of change of f at P occurs ~~at~~ the direction of $\nabla f(P) = \langle 4, 4 \rangle$

or $\frac{1}{\sqrt{32}} \langle 4, 4 \rangle$ (unit vector in the dir. of greatest increase of f at P).

Max rate of change of f at is $|\nabla f| = \sqrt{32}$.

4. (15 points) Find the absolute minimum and maximum values of the function $f(x, y) = 4x + 6y - x^2 - y^2$ on the region $D = \{(x, y) : 0 \le x \le 4, 0 \le y \le 5\}$.



(1) Crit. pts:

$f_x = 4 - 2x$

$\Rightarrow f_x = 0$ when $x = 2$

$f_y = 6 - 2y$

$\Rightarrow f_y = 0$ when $y = 3$

So $\boxed{(2,3)}$ is the only crit. pt. on the interior of D.

(2) Boundary:

on L_1 $(y=0)$, $f(x,y) = 4x - x^2$, which has a crit. pt. at $x = 2$, so we must consider $\boxed{(2,0)}$.

on L_2 $(x=4)$, $f(x,y) = \cancel{16}\, 6y - \cancel{16} - y^2 = 6y - y^2$, which has a crit. pt. at $y = 3$, so we must consider $\boxed{(4,3)}$.

on L_3 $(y=5)$, $f(x,y) = 4x + 30 - x^2 - 25 = -x^2 + 4x + 5$, which has a crit. pt. at $x = 2$, so we must consider $\boxed{(2,5)}$.

on L_4 $(x=0)$, $f(x,y) = 6y - y^2$, which has a crit. pt. at $y = 3$, so we must consider $\boxed{(0,3)}$.

Of course, we must also consider $\boxed{(0,0)}$, $\boxed{(0,5)}$, $\boxed{(4,0)}$, and $\boxed{(4,5)}$

(3) Compare values

$\boxed{f(2,3) = 13}$ abs. max.

$f(2,0) = 4$

$f(4,3) = 9$

$f(2,5) = 9$

$f(0,3) = 9$

$\boxed{f(0,0) = 0}$

$f(0,5) = 5$

$\boxed{f(4,0) = 0}$

abs. min.

$f(4,5) = 5$

So the abs. max. of f on D is 13, attained at (2,3), and the abs. min. of f is 0 attained at (0,0) and (4,0).

5. (14 points) Find the greatest and smallest values of $f(x, y) = 2xy$ on the ellipse $g(x,y) = \frac{x^2}{4} + y^2 = 1 = k$. $g(x,y)=k$ is closed & bounded, f cont's $\Rightarrow$ f

LM: attains ~~a~~ max and min o~~n~~ $g(x,y)=k$. These extrema of f along the level curve $g(x,y)=k$ will occur when the tangent line to $g(x,y)=k$ is parallel to the tangent line to a level curve of f, or when ∇f is $\parallel$ to ∇g, ie $\nabla f = \lambda \nabla g$.

$$\nabla f = \langle 2y, 2x \rangle \text{ and } \nabla g = \langle \tfrac{x}{2}, 2y \rangle$$

so $\nabla f = \lambda \nabla g \Rightarrow \begin{cases} 2y = \lambda \frac{x}{2} \\ 2x = \lambda 2y \end{cases}$ so we must solve $\begin{cases} ① \ 4y - \lambda x = 0 \\ ② \ 2x - 2\lambda y = 0 \\ ③ \ x^2 + 4y^2 = 4 \end{cases}$

Algebra: By ①, either $x=0$ or $\lambda = \frac{4y}{x}$. If $x=0$, by ③ we get $y = \pm 1$, so $\boxed{(0, \pm 1)}$

If $\lambda = \frac{4y}{x}$, by ② $2x - 2\left(\frac{4y}{x}\right)y = 0$, or $2x^2 - 8y^2 = 0$.

By ③, $2x^2 + 8y^2 = 8$, so we have $2x^2 - 8y^2 = 0$
and $2x^2 + 8y^2 = 8$

Adding, we get $4x^2 = 8$. Subtracting, we get $16y^2 = 8$

$x^2 = 2$ $\qquad$ $y^2 = \frac{1}{2}$

$\boxed{x = \pm\sqrt{2}}$ $\qquad$ $\boxed{y = \pm\sqrt{1/2}}$

Check values: $f(0, \pm 1) = 0$

$f(\pm\sqrt{2}, \pm\sqrt{1/2}) = 2 \leftarrow$ abs. max of f on $g=k$

$f(\pm\sqrt{2}, \mp\sqrt{1/2}) = -2 \leftarrow$ abs. min. of f on $g=k$

6. (14 points) Evaluate the double integral

$$\underline{I} = \int_0^1 \int_0^2 xye^{xy^2}\, dx\, dy.$$

$$\underline{I} = \int_0^2 \int_0^1 xy\, e^{xy^2}\, dy\, dx$$ by Fubini since $f(x,y) = xye^{xy^2}$ con'ts

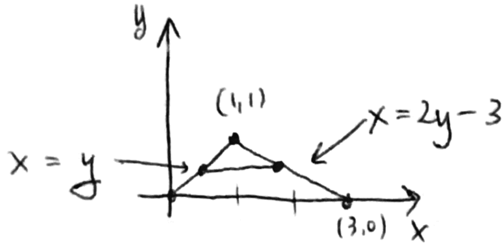
$$= \int_0^2 \left[\tfrac{1}{2} e^{xy^2}\right]_{y=0}^{y=1} dx = \int_0^2 \left(\tfrac{1}{2}e^0 - \tfrac{1}{2}e^x\right) dx = \tfrac{1}{2}x - \tfrac{1}{2}e^x \Big|_0^2$$

$u = xy^2$

$\frac{1}{2}du = xy\,dy$

$$= 1 - \tfrac{1}{2}e^2 - \left(0 - \tfrac{1}{2}\right) = \boxed{\tfrac{3}{2} - \tfrac{1}{2}e^2}$$

7. (14 points) Evaluate $I = \iint_R y\,dA$ where R is the triangular region with vertices $(0,0)$, $(3,0)$, and $(1,1)$.



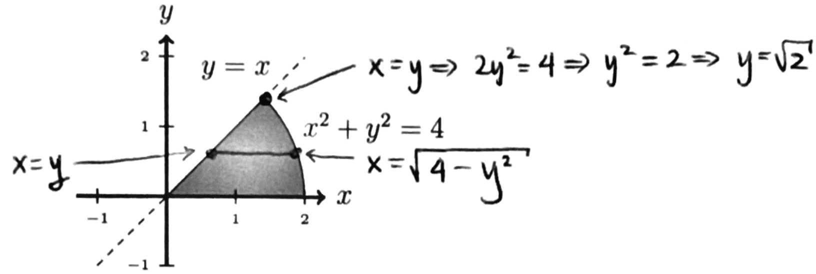
$$R = \{(x,y) \mid 0 \le y \le 1,\ y \le x \le 2y-3\}$$

$$I = \int_0^1 \int_y^{2y-3} y\,dx\,dy = \int_0^1 \left[xy\right]_y^{2y-3} dy$$

$$= \int_0^1 \left((2y-3)y - y^2\right) dy = \int_0^1 \left(y^2 - 3y\right) dy = \frac{1}{3}y^3 - \frac{3}{2}y^2 \Big|_0^1 = \frac{1}{3} - \frac{3}{2} = \boxed{-\frac{7}{6}}$$

$$\boxed{I = -\frac{7}{6}}$$

8. (15 points) Consider the integral $I = \iint_R (x^2+y^2)^4\,dA$ where R is the region above the line $y=0$, below the line $y=x$, and inside the circle $x^2+y^2=4$, as pictured below:



(a) Set up (no need to evaluate) an iterated double integral for I in rectangular xy-coordinates with your choice of integration order.

$$R = \{(x,y) \mid 0 \le y \le \sqrt{2},\ y \le x \le \sqrt{4-y^2}\}$$

$$I = \int_0^{\sqrt{2}}\int_y^{\sqrt{4-y^2}} (x^2+y^2)^4\,dx\,dy$$

(b) Evaluate the double integral in polar $r\theta$-coordinates.

$$R = \{(r,\theta) \mid 0 \le r \le 2,\ 0 \le \theta \le \tfrac{\pi}{4}\}$$

$$f(r\cos\theta, r\sin\theta) = (r^2)^4 = r^8$$

$$dA = r\,dr\,d\theta$$

$$I = \int_0^{\pi/4}\int_0^2 r^8\, r\,dr\,d\theta = \int_0^{\pi/4} \tfrac{1}{10} r^{10}\Big|_0^2\,d\theta = \frac{2^{10}}{10}\,\pi/4 = \boxed{\frac{2^{10}\pi}{40}}$$