Math 162 - Spring 2025
Workshop 9
March 31 - April 4
Calculus of Polar Curves and Sequences

Problem 1. Sketch the curve $r = 1 + \cos(\theta)$ and find the length of the curve.

Solution: Looking at the graph, we see the bounds are 0 and 2π. We have $\dfrac{dr}{d\theta} = -\sin\theta$, so using the formula $L = \int_a^b \sqrt{r^2 + (\frac{dr}{d\theta})^2} d\theta$, we get

$$\begin{aligned} L &= \int_0^{2\pi} \sqrt{(1+\cos\theta)^2 + (-\sin\theta)^2} d\theta = \int_0^{2\pi} \sqrt{2 + 2\cos\theta} \\ &= \int_0^{2\pi} 2 \left|\cos(\frac{\theta}{2})\right| d\theta = 2\int_0^{\pi} 2\cos(\frac{\theta}{2}) = 8\sin(\frac{\theta}{2}) \mid_0^{\pi} = 8 \end{aligned}$$

where we use $1 + \cos(2\pi)2 = \cos^2\theta$ to get $1 + \cos(\theta) = 2\cos^2(\frac{\theta}{2})$.

Problem 2. Sketch the five petaled rose $r = \cos(5\theta)$. Find the area of the region bounded by one loop of this curve.

Solution: Use the formula $A = \displaystyle\int_a^b \frac{1}{2} r^2 d\theta$. One choice of endpoints is $-\dfrac{\pi}{10}$ and $\dfrac{\pi}{10}$. Using the formula, we obtain

$$\begin{aligned} \int_{-\frac{\pi}{10}}^{\frac{\pi}{10}} \frac{1}{2}\cos^2(5\theta) d\theta &= \frac{1}{2}\int_{-\frac{\pi}{10}}^{\frac{\pi}{10}} \frac{1 + \cos(10\theta)}{2} d\theta \\ &= \frac{1}{2}\left(\frac{\theta}{2} + \frac{\sin(10\theta)}{20}\right) = \frac{\pi}{20} \end{aligned}$$

Problem 3. For each of the following sequences, write out the first four terms. Then determine if the sequences converge and diverge. If they converge, find the limit.

1. $a_n = \dfrac{3 + 5n^2}{n + n^2}$
2. $\{\dfrac{(2n-1)!}{(2n+1)!}\}$
3. $\{n^2 e^{-n}\}$

Solution:

1. $a_1 = 4$, $a_2 = \frac{23}{6}$, $a_3 = 4$ and $a_4 = \dfrac{83}{20}$. $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{5n^2}{n^2} = 5$

2. The trick is to rewrite $(2n+1)! = (2n+1)(2n)(2n-1)!$ so that

$$a_n = \frac{(2n-1)!}{(2n+1)!} = \frac{(2n-1)!}{(2n+1)(2n)(2n-1)!} = \frac{1}{(2n+1)(2n)}$$

   The first four terms are $a_1 = \dfrac{1}{6}, a_2 = \dfrac{1}{20}, a_3 = \dfrac{1}{42}, a_4 = \dfrac{1}{72}$. The limit is

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{(2n+1)(2n)} = 0$$

3. The first four terms are $a_1 = e^{-1}, a_2 = 4e^{-2}, a_3 = 9e^{-3}, a_4 = 16e^{-4}$. The limit requires two applications of L'Hopital's rule

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{2n}{e^n} = \lim_{n\to\infty} \frac{2}{e^n} = 0$$

Problem 4. Define the sequence a_n by $a_1 = \sqrt{2}$ and recursively by $a_{n+1} = \sqrt{2 + a_n}$ for all positive integers n.

1. Use induction to show that a_n is a monotonically increasing sequence.
2. Show that a_n is bounded by showing that $a_n > 0$ and $a_n < 3$ for all $n > 0$.
3. By the Monotone Sequence Theorem, we know that a_n converges. Find the limit of a_n.

Solution:

1. First note that $a_2 = \sqrt{\sqrt{2}+2} > \sqrt{2} = a_1$. The induction argument is to derive $a_{n+1} > a_n$ from $a_n > a_{n-1}$, where the initial step $a_2 > a_1$ is already observed. This would imply that the sequence is monotonically increasing. To check the induction step, note that $a_n > a_{n-1}$ implies $\sqrt{a_n + 2} > \sqrt{a_{n-1} + 2}$, i.e. $a_{n+1} > a_n$, as desired.

2. By the above, we know $a_n > a_0 = \sqrt{2} > 0$. We induct on the statement $a_n < 3$ to show it is true for all n. The base case is $n = 1$, and $a_n = \sqrt{2} < 3$. Suppose $a_n < 3$. We show $a_{n+1} < 3$. By the recurrence relation

$$a_{n+1} = \sqrt{2 + a_n} < \sqrt{2+3} = \sqrt{5} < 3$$

   which completes the inductive step.

3. Let $L = \lim_{n\to\infty} a_n$. By taking limits of the recurrence relation, we get

$$\lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \sqrt{a_n + 2} = \sqrt{\lim_{n\to\infty} a_n + 2}$$

so $L = \sqrt{L+2}$. Squaring and moving everything to one side we get $L^2 - L - 2 = 0$. This factors as $(L-2)(L+1) = 0$ so either $L = 2$ or $L = -1$. Since $a_n > 0$, $L \geq 0$, so $L = 2$.