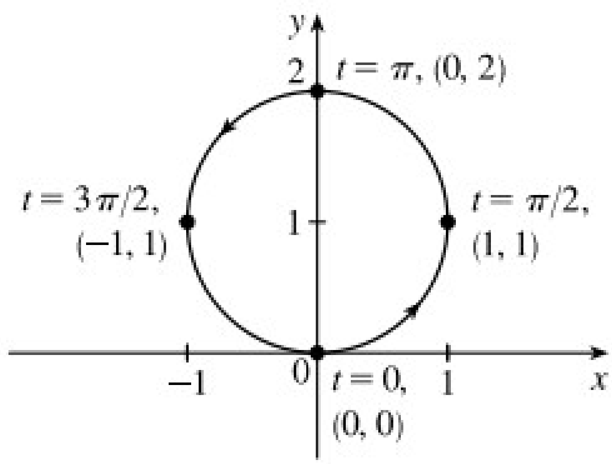
Math 162 - Spring 2025
Workshop 8 Solution
Mar 24 - Mar 28
10.1-4 Parametric Equations and Polar Coordinates

Problem 1. $x = sin(t), \quad y = 1 - cos(t), \quad 0 \leq t \leq 2\pi$
(a) Sketch the curve by using the parametric equations to plot points. Indicate with arrow the direction in which the curve is traced as t increases.
(b) Eliminate the parameter to find a Cartesian equation of the curve.
Solution:

$$x = \sin t, \quad y = 1 - \cos t, \quad 0 \leq t \leq 2\pi$$

(a)

t	0	$\pi/2$	π	$3\pi/2$	2π
x	0	1	0	-1	0
y	0	1	2	1	0



(b) $x = \sin t, y = 1 - \cos t \quad$ [or $y - 1 = -\cos t$] $\Rightarrow$

$$x^2 + (y-1)^2 = (\sin t)^2 + (-\cos t)^2 \Rightarrow x^2 + (y-1)^2 = 1.$$

As t varies from 0 to 2π, the circle with center $(0, 1)$ and radius 1 is traced out.

Problem 2. Find parametric equations for the path of a particle that moves along the circle $x^2 + (y-1)^2 = 4$ in the manner described.
(a) Once around clockwise, starting at $(2, 1)$
(b) Three times around counterclockwise, starting at $(2, 1)$
(c) Halfway around counterclockwise, starting at $(0, 3)$

Solution:
The circle $x^2+(y-1)^2=4$ has center $(0,1)$ and radius 2 , then it can be represented by $x=2\cos t$, $y=1+2\sin t, 0\leq t\leq 2\pi$. This representation gives us the circle with a counterclockwise orientation starting at $(2,1)$.
(a) To get a clockwise orientation, we could change the equations to $x=2\cos t, y=1-2\sin t, 0\leq t\leq 2\pi$.
(b) To get three times around in the counterclockwise direction, we use the original equations $x=2\cos t, y=1+2\sin t$ with the domain expanded to $0\leq t\leq 6\pi$.
(c) To start at $(0,3)$ using the original equations, we must have $x_1=0$; that is, $2\cos t=0$. Hence, $t=\frac{\pi}{2}$. So we use $x=2\cos t, y=1+2\sin t, \frac{\pi}{2}\leq t\leq\frac{3\pi}{2}$.

Alternatively, if we want t to start at 0 , we could change the equations of the curve. For example, we could use $x=-2\sin t, y=1+2\cos t, 0\leq t\leq\pi$.

Problem 3. $x=t-ln(t), \quad y=t+ln(t)$
Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$. For which values of t is curve concave upward?

Solution:
$x=t-\ln t, y=t+\ln t \quad$ [note that $t>0$] $\Rightarrow \frac{dy}{dx}=\frac{dy/dt}{dx/dt}=\frac{1+1/t}{1-1/t}=\frac{t+1}{t-1}\Rightarrow\frac{d^2y}{dx^2}=\frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{dx/dt}=\frac{\frac{(t-1)(1)-(t+1)(1)}{(t-1)^2}}{(t-1)/t}=\frac{-2t}{(t-1)^3}$. The curve is CU when $\frac{d^2y}{dx^2}>0$, that is, when $0<t<1$.

Problem 4. Identify the curve by finding a Cartesian equation for the curve
(a) $r=4sec(\theta)$, (b) $r^2sin(2\theta)=1$ (c) $r=5cos(\theta)$
Solution:
(a) $r=4\sec\theta\Leftrightarrow\frac{r}{\sec\theta}=4\Leftrightarrow r\cos\theta=4\Leftrightarrow x=4$, a vertical line.
(b)

$$r^2\sin 2\theta=1\Leftrightarrow r^2(2\sin\theta\cos\theta)=1\Leftrightarrow 2(r\cos\theta)(r\sin\theta)=1\Leftrightarrow 2xy=1\Leftrightarrow xy=\frac{1}{2},$$ a hyperbola

centered at the origin with foci on the line $y=x$.
(c)

$$r=5\cos\theta\Rightarrow r^2=5r\cos\theta\Leftrightarrow x^2+y^2=5x\Leftrightarrow x^2-5x+\frac{25}{4}+y^2=\frac{25}{4}\Leftrightarrow\left(x-\frac{5}{2}\right)^2+y^2=\frac{25}{4},$$

a circle of radius $\frac{5}{2}$ centered at $\left(\frac{5}{2},0\right)$. The first two equations are actually equivalent since $r^2=5r\cos\theta\Rightarrow r(r-5\cos\theta)=0\Rightarrow r=0$ or $r=5\cos\theta$. But $r=5\cos\theta$ gives the point $r=0$ (the pole) when $\theta=0$. Thus, the equation $r=5\cos\theta$ is equivalent to the compound condition ($r=0$ or $r=5\cos\theta$).

Problem 5. Find a polar equation for the curve represented by the given Cartesian equation.
(a) $y=\sqrt{3}x$, (b) $y=-2x^2$, , (c) $x^2+y^2=4y$

Solution:
(a) $y=\sqrt{3}x \Rightarrow \frac{y}{x}=\sqrt{3}[x\neq 0] \Rightarrow \tan\theta=\sqrt{3} \Rightarrow \theta=\frac{\pi}{3}$ or $\frac{4\pi}{3}$ [either incudes the pole]
(b)

$$y=-2x^2 \Rightarrow r\sin\theta=-2(r\cos\theta)^2 \Rightarrow r\sin\theta+2r^2\cos^2\theta=0 \Rightarrow r\left(\sin\theta+2r\cos^2\theta\right)=0 \Rightarrow$$

$r=0$ or $r=-\frac{\sin\theta}{2\cos^2\theta}=-\frac{1}{2}\tan\theta\sec\theta$. $r=0$ is included in $r=-\frac{1}{2}\tan\theta\sec\theta$ when $\theta=0$, so the curve is represented by the single equation $r=-\frac{1}{2}\tan\theta\sec\theta$.
(c)

$$x^2+y^2=4y \Rightarrow r^2=4r\sin\theta \Rightarrow r^2-4r\sin\theta=0 \Rightarrow r(r-4\sin\theta)=0 \Rightarrow r=0 \text{ or } r=4\sin\theta.$$

$r=0$ is included in $r=4\sin\theta$ when $\theta=0$, so the curve is represented by the single equation $r=4\sin\theta$.

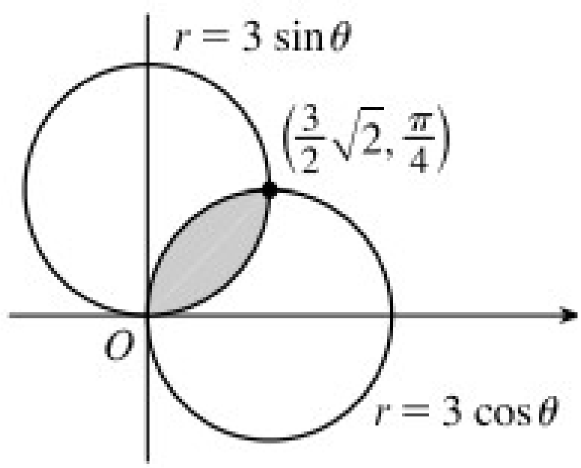
Problem 6. Find the area of the region that lies inside both curves
(a) $r=3sin(\theta), \quad r=3cos(\theta)$
(b) $r=1+cos(\theta), \quad ,r=1-cos(\theta)$

Solution:

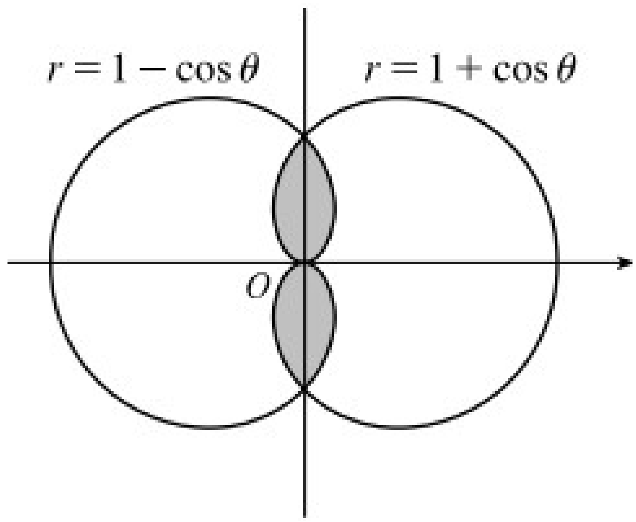
$$3\sin\theta=3\cos\theta \Rightarrow \frac{3\sin\theta}{3\cos\theta}=1 \Rightarrow \tan\theta=1 \Rightarrow \theta=\frac{\pi}{4} \Rightarrow$$

(a)

$$\begin{aligned} A &= 2\int_0^{\pi/4}\frac{1}{2}(3\sin\theta)^2 d\theta=\int_0^{\pi/4} 9\sin^2\theta d\theta=\int_0^{\pi/4} 9\cdot\frac{1}{2}(1-\cos 2\theta)d\theta \\ &= \int_0^{\pi/4}\left(\frac{9}{2}-\frac{9}{2}\cos 2\theta\right)d\theta=\left[\frac{9}{2}\theta-\frac{9}{4}\sin 2\theta\right]_0^{\pi/4}=\left(\frac{9\pi}{8}-\frac{9}{4}\right)-(0-0) \\ &= \frac{9\pi}{8}-\frac{9}{4} \end{aligned}.$$



(b)
$$
\begin{aligned}
A &= 4\int_0^{\pi/2} \frac{1}{2}(1-\cos\theta)^2 d\theta = 2\int_0^{\pi/2} \left(1 - 2\cos\theta + \cos^2\theta\right) d\theta \\
&= 2\int_0^{\pi/2} \left[1 - 2\cos\theta + \frac{1}{2}(1+\cos 2\theta)\right] d\theta \\
&= 2\int_0^{\pi/2} \left(\frac{3}{2} - 2\cos\theta + \frac{1}{2}\cos 2\theta\right) d\theta = \int_0^{\pi/2} (3 - 4\cos\theta + \cos 2\theta) d\theta \\
&= \left[3\theta - 4\sin\theta + \frac{1}{2}\sin 2\theta\right]_0^{\pi/2} = \frac{3\pi}{2} - 4
\end{aligned}
$$



Problem 7. Find the exact length of the polar curve.
(a) $r = e^{\theta/2}, \quad 0 \le \theta \le \pi/2$
(b) $r = 2(1 + cos(\theta))$

Solution:

(a)
$$
\begin{aligned}
L &= \int_0^{\pi/2} \sqrt{r^2 + (dr/d\theta)^2} d\theta = \int_0^{\pi/2} \sqrt{\left(e^{\theta/2}\right)^2 + \left(\frac{1}{2}e^{\theta/2}\right)^2} d\theta = \int_0^{\pi/2} \sqrt{\left(e^{\theta/2}\right)^2\left(1 + \frac{1}{4}\right)} d\theta \\
&= \sqrt{\frac{5}{4}} \int_0^{\pi/2} \left|e^{\theta/2}\right| d\theta = \frac{\sqrt{5}}{2}\int_0^{\pi/2} e^{\theta/2} d\theta = \frac{\sqrt{5}}{2}\left[2e^{\theta/2}\right]_0^{\pi/2} = \sqrt{5}\left(e^{\pi/4} - 1\right)
\end{aligned}
$$

(b)
$$
\begin{aligned}
L &= \int_a^b \sqrt{r^2 + (dr/d\theta)^2} d\theta = \int_0^{2\pi} \sqrt{[2(1+\cos\theta)]^2 + (-2\sin\theta)^2} d\theta = \int_0^{2\pi} \sqrt{4 + 8\cos\theta + 4\cos^2\theta + 4 \\
&= \int_0^{2\pi} \sqrt{8 + 8\cos\theta} d\theta = \sqrt{8}\int_0^{2\pi} \sqrt{1+\cos\theta} d\theta = \sqrt{8}\int_0^{2\pi} \sqrt{2 \cdot \frac{1}{2}(1+\cos\theta)} d\theta \\
&= \sqrt{8}\int_0^{2\pi} \sqrt{2\cos^2\frac{\theta}{2}} d\theta = \sqrt{8}\sqrt{2}\int_0^{2\pi} \left|\cos\frac{\theta}{2}\right| d\theta = 4 \cdot 2\int_0^{\pi} \cos\frac{\theta}{2} d\theta \quad \text{[by symmetry]} \\
&= 8\left[2\sin\frac{\theta}{2}\right]_0^{\pi} = 8(2) = 16
\end{aligned}
$$