Math 162 - Spring 2025
Workshop 7
March 17 - March 21
Area of a surface of revolution - Parametric curves

Problem 1.

(a) Compute the area of surface of revolution obtained by rotating the curve $y = \sqrt{4 - x^2}$ around the x-axis.

(b) Do the same for the curve $y = 1 - |x|$, $-1 \leq x \leq 1$.

Answer:

(a)

$$\begin{aligned} A &= 2\pi \int_{-2}^{2} y\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx \\ &= 2\pi \int_{-2}^{2} \sqrt{4 - x^2}\sqrt{1 + \frac{x^2}{4 - x^2}}\, dx \\ &= 2\pi \int_{-2}^{2} 2\, dx = 16\pi. \end{aligned}$$

(b)

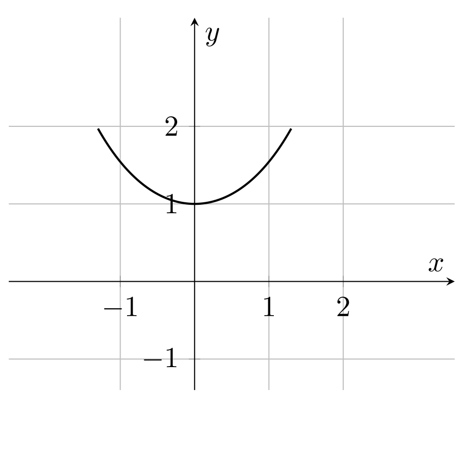
$$\begin{aligned} A &= 2\pi \int_{-1}^{0} (1 + x)\sqrt{1 + 1}\, dx + 2\pi \int_{0}^{1} (1 - x)\sqrt{1 + 1}\, dx \\ &= 2\pi\sqrt{2}\left(\left[x + \frac{x^2}{2}\right]_{-1}^{0} + \left[x - \frac{x^2}{2}\right]_{0}^{1}\right) \\ &= 2\pi\sqrt{2}\left(\frac{1}{2} + \frac{1}{2}\right) = 2\pi\sqrt{2}. \end{aligned}$$

□

Problem 2. The curve

$$y = \frac{e^x + e^{-x}}{2}, \quad -1 \leq x \leq 1,$$

is graphed below. Find the surface area of the solid of revolution obtained by rotating this curve about the x-axis.



Answer:
We have

$$\frac{dy}{dx} = \frac{e^x - e^{-x}}{2}$$

so that

$$\begin{aligned}
1 + \left(\frac{dy}{dx}\right)^2 &= 1 + \frac{(e^x)^2 - 2e^x e^{-x} + (e^{-x})^2}{4} \\
&= \frac{4 + e^{2x} - 2 + e^{-2x}}{4} \\
&= \frac{e^{2x} + 2 + e^{-2x}}{4} \\
&= \left(\frac{e^x + e^{-x}}{2}\right)^2 = y^2.
\end{aligned}$$

Therefore, as we are rotating about the x-axis and both y, $\sqrt{1 + (dy/dx)^2}$ are even functions of x, we have

$$\begin{aligned}
S &= \int_{-1}^{1} 2\pi y \sqrt{1 + (dy/dx)^2}\, dx \\
&= 2\int_0^1 2\pi y^2\, dx \\
&= 4\pi \int_0^1 \frac{1}{4}\left(e^{2x} + 2 + e^{-2x}\right) dx \\
&= \pi \int_0^1 \left(\frac{1}{2}e^{2x} + 2 - \frac{1}{2}e^{-2x}\right) dx \\
&= \pi \left[\frac{1}{2}e^{2x} + 2x - \frac{1}{2}e^{-2x}\right]_{x=0}^{x=1} \\
&= \pi \left(\frac{1}{2}e^2 + 2 - \frac{1}{2}e^{-2}\right).
\end{aligned}$$

□

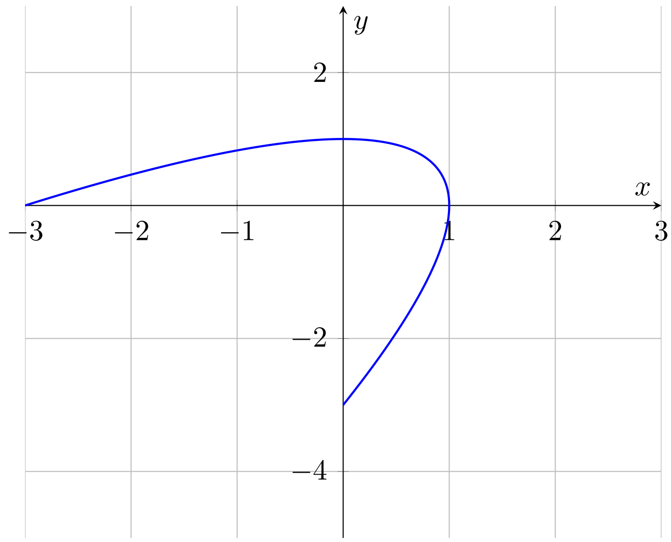
Problem 3. Sketch the curve by using the parametric equations to plot points. Indicate with an arrow the direction in which the curve is traced as t increases.

a) $x = 1 - t^2, \quad y = 2t - t^2, \quad -1 \le t \le 2.$

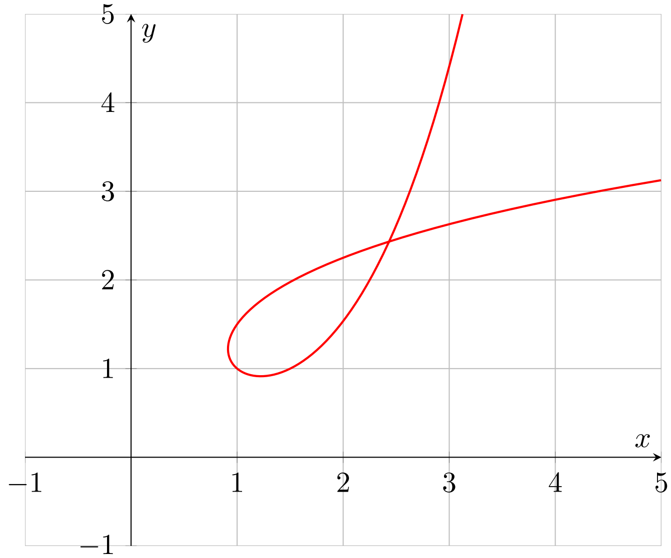
b) $x = 2^t - t, \quad y = 2^{-t} + t, \quad -3 \le t \le 3.$

Answer.

a) First curve: $x = 1 - t^2, y = 2t - t^2$



b) Second curve: $x = 2^t - t, y = 2^{-t} + t$



□

Problem 4.

1. Eliminate the parameter to find a Cartesian equation of the curve.

2. Sketch the curve and indicate with an arrow the direction in which the curve is traced as the parameter increases.

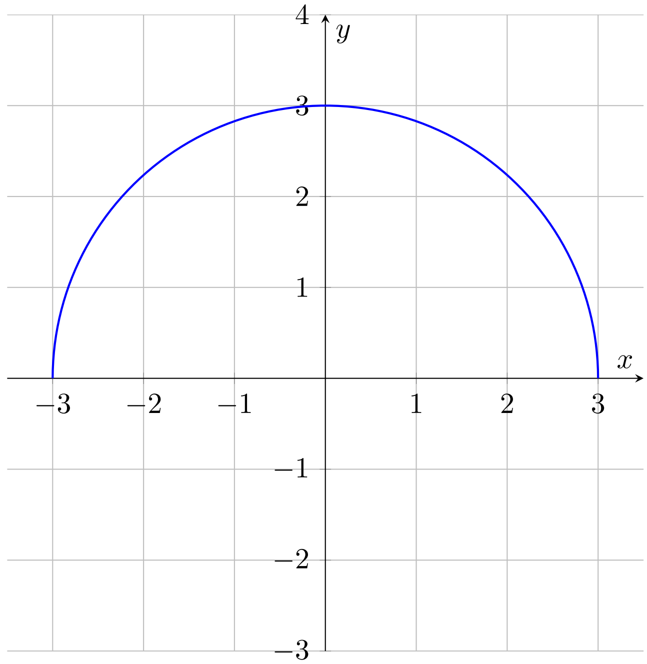
a) $x = 3\cos t, \quad y = 3\sin t, \quad 0 \leq t \leq \pi.$

b) $x = \cos\theta, \quad y = \sec^2\theta, \quad 0 \leq \theta < \frac{\pi}{2}.$

Answer.

a) $x^2 + y^2 = 9$

$x = 3\cos t, y = 3\sin t$



b) $x^2 y = 1$.

$x = \cos\theta, y = \sec^2\theta$

□

Problem 5. Find parametric equations for the position of a particle moving along a circle as described.

The particle travels clockwise around a circle centered at the origin with radius 5 and completes a revolution in 4πseconds.

Answer.

The standard parametric equations for counterclockwise motion are:

$$x = r\cos\theta, \quad y = r\sin\theta$$

For clockwise motion, we use

$$x = r\cos\theta, \quad y = r\sin(-\theta)$$

The radius $r = 5$, the particle completes **one revolution in** 4π seconds, the angular velocity ω is:

$$\omega = \frac{2\pi}{4\pi} = \frac{1}{2}\text{ rad/sec}$$

Thus

$$x = 5\cos\left(\frac{t}{2}\right), \quad y = -5\sin\left(\frac{t}{2}\right), 0 \le t \le 2\pi.$$

□

Problem 6. Find parametric equations to represent the line segment from $(-2, 7)$ to $(3, -1)$.

Answer. Using the parametric formula:

$$x = x_1 + t(x_2 - x_1), \quad y = y_1 + t(y_2 - y_1), \quad 0 \le t \le 1,$$

we obtain:

$$x = -2 + 5t$$

$$y = 7 - 8t$$

where $0 \le t \le 1$. □

Problem 7. The position of a red particle at time t is given by

$$x = t + 5, \quad y = t^2 + 4t + 6$$

and the position of a blue particle is given by

$$x = 2t + 1, \quad y = 2t + 6.$$

a) Graph the paths of both particles. At how many points do the graphs intersect? Do the particles collide? If so, find the collision points.

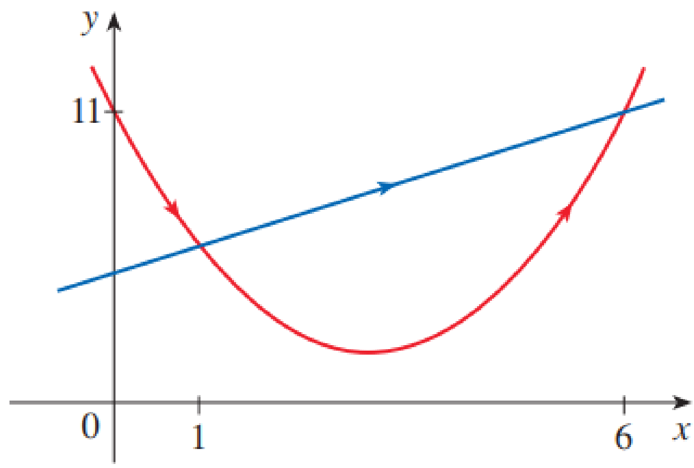
b) If the equations of the blue particles are

$$x = 2t + 4, \quad y = 2t + 9,$$

then does red and blue particles collide?

Answer:

a) Their paths are shown in the graph. The path of the red particle is the graph of $y = x^2 - 6x + 11$, and the path of the blue particle is $y = x + 5$.



Solving $x^2 - 6x + 11 = x + 5$, we have the intersection points $(6, 11), (1, 6)$.
Set the position vectors of the two particles equal to each other and solve for the time variable

$$t + 5 = 2t + 1, \quad t^2 + 4t + 6 = 2t + 6.$$

There is no solution, so there is no collision point.
b) Solve

$$t + 5 = 2t + 4, t^2 + 4t + 6 = 2t + 9,$$

we have $t = 1$. Thus, the collision point is $(6, 11)$. □