Math 162 - Spring 2025
Workshop 6 Solutions
March 3 - March 7
Indefinite Integrals and Arclength

Problem 1. Evaluate the following improper integrals.

1. $\int_2^{\infty} \frac{1}{x^2+2x-3} dx$
2. $\int_0^{1} \frac{1}{x^2+2x-3} dx$

Solution: First, we compute the antiderivative of $\frac{1}{x^2+2x-3} = \frac{1}{(x-1)(x+3)}$ using partial fractions. We have $\frac{1}{x^2+2x-3} = \frac{1}{4(x-1)} - \frac{1}{4(x+3)}$, so $\int \frac{1}{x^2+2x+3} = \frac{1}{4}\ln|x-1| - \frac{1}{4}\ln\left|\frac{x-1}{x-4}\right|$.

For (a), the function is continuous on the interval $[2,\infty)$, so we just need to take limits at infinity.

$$\int_2^{\infty} \frac{1}{x^2+2x-3} = \lim_{t\to\infty}\int_2^t \frac{1}{x^2+2x-3} = \lim_{t\to\infty}\frac{1}{4}\ln\left|\frac{x-1}{x-4}\right| |_2^t$$
$$= \lim_{t\to\infty}\frac{1}{4}\ln\left|\frac{t-1}{t+3}\right| - \frac{1}{4}\ln\frac{1}{5} = \frac{1}{4}\ln|1| - \frac{1}{4}\ln\frac{1}{5} = -\frac{1}{4}\ln\frac{1}{5}$$

For (b), the function has an infinite discontinuity at $x=1$, so

$$\int_0^1 \frac{1}{x^2+2x-3} = \lim_{t\to1^-}\int_0^t \frac{1}{x^2+2x-3} = \lim_{t\to1^-}\frac{1}{4}\ln\left|\frac{t-1}{t+3}\right| - \frac{1}{4}\ln\left|-\frac{1}{3}\right| = -\infty$$

because $\frac{t-1}{t+3}$ approaches 0 as $t\to1^-$, and $\ln(x)$ approaches $-\infty$ as $x\to0$.

Problem 2. For what p does $\int_0^1 x^p \ln x dx$ converge?

Solution: As usual, split into the case $p=-1$ and $p\neq-1$. If $p=-1$, then this improper integral is

$$\int_0^1 \frac{\ln x}{x} = \lim_{t\to0^+}\frac{(\ln|x|)^2}{2} |_t^1 = \lim_{t\to0^+} -\frac{(\ln|t|)^2}{2},$$

which diverges.

If $p \neq -1$, then integration by parts gives us with $u = \ln x$, $du = \dfrac{dx}{x}$ and $v = \frac{x^{p+1}}{p+1}$ and $dv = x^p dx$.

$$\int_0^1 x^p \ln x dx = \lim_{t \to 0^+} \frac{x^{p+1} \ln x}{p+1} \,|_t^1 - \int_t^1 \frac{x^p}{p+1} dx = \lim_{t \to 0^+} \frac{(\ln x) x^{p+1}}{p+1} - \frac{x^{p+1}}{(p+1)^2} \,|_t^1$$

This integral converges if the limit exists. So we are left to find the p where $\lim_{t \to 0^+} \dfrac{t^{p+1} \ln t}{p+1}$ converges (why?).

$$\lim_{t \to 0^+} (\frac{\ln t}{p+1}) t^{p+1} = \lim_{t \to 0^+} \frac{\frac{\ln t}{p+1}}{t^{-(p+1)}} = \lim_{t \to 0^+} \frac{\frac{1}{(p+1)t}}{-(p+1)t^{-(p+2)}} = \lim_{t \to 0^+} \frac{-1}{(p+1)^2} t^{p+1}$$

This converges if $p + 1 \geq 0$. Considering that we ruled out $p = -1$, we only have convergence for $p > -1$.

Problem 3. Find the arclength of the curve $y = \dfrac{1}{4}x^2 - \dfrac{1}{2}\ln x$ on the interval $1 \leq x \leq 2$.

Solution: We need to compute y' first, $y' = \frac{x}{2} - \frac{1}{2x}$. Using the formula for arclength, we get

$$L = \int_1^2 \sqrt{1 + (\frac{x}{2} - \frac{1}{2x})^2} dx = \sqrt{1 + x^2 - \frac{1}{2} + \frac{1}{4x^2}} dx = \int_1^2 \sqrt{\frac{x^2}{4} + \frac{1}{2} + \frac{1}{4x}} dx$$
$$= \int_1^2 \sqrt{(\frac{x}{2} + \frac{1}{2x})^2} dx = \int_1^2 \frac{x}{2} + \frac{1}{2x} dx = \frac{x^2}{4} + \frac{1}{2} \ln x \,|_1^2 = 1 + \frac{1}{2}\ln 2 - \frac{1}{4} = \frac{3}{4} + \frac{1}{2}\ln 2$$

Problem 4. Find the arclength of the curve $x = \ln(\cos y)$ on the interval $0 \leq y \leq \frac{\pi}{3}$.

Solution: First find x', $x' = \dfrac{1}{\cos y}(-\sin y) = -\tan y$. Using the arclength formula, we get

$$L = \int_0^{\pi/3} \sqrt{1 + \tan^2 y} dy = \int_0^{\pi/3} \sec y dy = \ln|\sec y + \tan y| \,|_0^{\pi/3} = \ln\left|2 + \sqrt{3}\right| - \ln|1| = \ln(2 + \sqrt{3})$$