Math 162 - Spring 2025
Workshop 4
Feb 10 - Feb 14
More work, Integration by parts, Trig integrals

Problem 1. A cable with mass 0.5kg/meter is lifting a load of 150 kg that is initially at the bottom of a 50-meter shaft. How much work is required to lift the load $\frac{1}{4}$ of the way up the shaft?

Denote $x = 0$ at the bottom and $x = 50$ at the top of the shaft. When the load is lifted $x-$meters, there will be $50 - x$ meters of chain left. Thus the total mass of the chain left is

$$\frac{1}{2}(50 - x) + 150 = 175 - \frac{1}{2}x(kg).$$

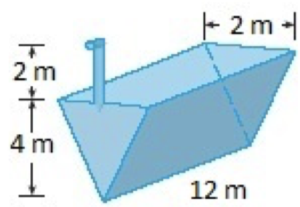
Thus the force required to hold the chain when the load is lifted x meters is

$$F(x) = 9.8(175 - \frac{1}{2}x) = 1715 - 4.9x.$$

The work is then

$$W = \int_0^{50/4} (1715 - 4.9x)dx \approx 21054(J).$$

Problem 2. The tank depicted below is full of water.

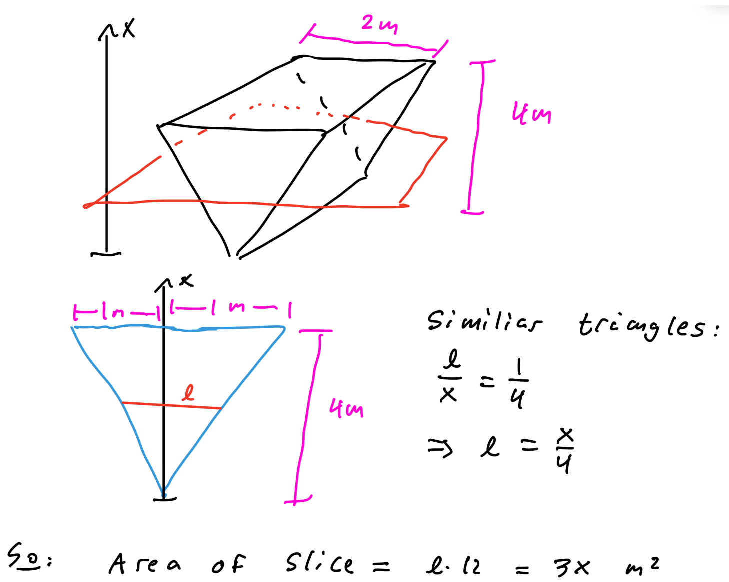


prism tank.png

1. Find an integral expression for the work required to pump the water out of the spout, given that the density of water is 1000 kilograms per meter cubed.

   Solution. Place an x-axis with the origin at the base of the tank and so that the positive direction is going towards the top of the tank. A horizontal slice of the tank x meters above the ground is a rectangle. Our first job is to determine the area of such a slice as a function of x.

   The long side of the rectangle is the length of the tank, which is 12 meters. The length of the shorter side can be found using similar triangles:



triangle tank.png

Note: The equation is WRONG in the picture (the idea is correct). Similar triangles gives

$$\frac{\ell}{x} = \frac{2}{4}$$

So that

$$\ell = \frac{x}{2}.$$

Thus the area of a slice is $6x$ meters squared. Now the weight of each slice is:

$$\begin{aligned} \text{weight} &= \text{mass} g \\ &= (\text{density})(\text{volume}) g \\ &= 1000 g \text{area } dx \\ &= 1000 g 6x \, dx, \end{aligned}$$

where $g \approx 9.8 \quad m/s^2$ is the acceleration due to gravity. Each slice is to be lifted from x meters above the bottom of the tank to 6 meters above the tank. Thus the distance each slice travels is $6 - x$ meters. All of the slices are between 0 and 4 meters above the floor of the tank. Therefore:

$$W = \int_0^4 (\text{distance})(\text{force}) = \int_0^4 (6 - x) 1000 g 6x \, dx = 6000 g \int_0^4 (6 - x) x \, dx.$$

2. How does your integral expression for the work change if the tank is only filled to a depth of 3 meters ?

   Solution. The distance a slice travels hasn't changed, and neither has the weight of each slice. What has changed is there are now no layers of water between 3 and 4 meters. So

$$W = 6000 g \int_0^3 (6 - x) x \, dx$$

3. How does your integral expression for the work change if the spout is 5 meters above the tank?

Solution. The tank of water is still full, and the weight of the layers of water is unchanged. What has changed is that each slice now travels a distance of $4+5-x=9-x$ meters rather than $6-x$ meters. So

$$W = 6000g\int_0^4 (9-x)x\,dx.$$

Problem 3. Evaluate the following integrals.

1. $\int_1^e \frac{\ln x}{x^2}dx$.
Put $u=\ln x$, $dv=x^{-2}dx$, by integration by parts,

$$\int_1^e \frac{\ln x}{x^2}dx = -\frac{\ln x}{x}\Big|_1^e + \int_1^e \frac{1}{x^2}dx = 1-\frac{2}{e}.$$

2. $\int(\ln x)^2dx$.
Put $u=(\ln x)^2$, $dv=dx$, by integration by parts,

$$\int(\ln x)^2dx = x(\ln x)^2 - 2\int \ln x dx.$$

Apply another integration by parts with $u=\ln x$, $dv=dx$, we have

$$\int \ln x dx = x\ln x - x + C.$$

Thus

$$\int(\ln x)^2dx = x(\ln x)62 - 2x\ln x + 2x$$

3. $\int e^{2x}\cos(\pi x-\frac{\pi}{3})dx$.
Integration by parts with $u=\cos(\pi x-\frac{\pi}{3})$, $dv=e^{2x}dx$, we have

$$I=\int e^{2x}\cos(\pi x-\frac{\pi}{3})dx = \frac{1}{2}e^{2x}\cos(\pi x-\frac{\pi}{3}) + \frac{\pi}{2}\int e^{2x}\sin(\pi x-\frac{\pi}{3})dx. \tag{1}$$

Apply another integration by parts with $u=\sin(\pi x-\pi/3)$,$dv=e^{2x}dx$, we have

$$\begin{aligned}\int e^{2x}\sin(\pi x-\frac{\pi}{3})dx &= \frac{1}{2}e^{2x}\sin(\pi x-\pi/3) - \frac{\pi}{2}\int e^{2x}\cos(\pi x-\pi/3)\\ &= \frac{1}{2}e^{2x}\sin(\pi x-\pi/3) - \frac{\pi}{2}I.\end{aligned}$$

Thus

$$I = \frac{1}{2}e^{2x}\cos(\pi x-\frac{\pi}{3}) + \frac{\pi}{2}\left(\frac{1}{2}e^{2x}\sin(\pi x-\pi/3) - \frac{\pi}{2}I\right).$$

Problem 4. Evaluate the following integrals.

1. $\int e^{\sqrt{x}}dx$. First use substitution $t = \sqrt{x}$, we write

$$\int e^{\sqrt{x}}dx = 2\int e^t t dt.$$

Apply integration by parts, $u = t$, $dv = e^t dt$, we have

$$\int e^t t dt = te^t - \int e^t dt = te^t - e^t + C.$$

Thus

$$\int e^{\sqrt{x}}dx = 2\sqrt{x}e^{\sqrt{x}} - 2e^{\sqrt{x}} + C.$$

2. $\int \cos(\ln x)dx$. Use substitution $t = \ln x$, we have

$$\int \cos(\ln x)dx = \int e^t \cos t dt.$$

Apply integration by parts twice, we have

$$\int e^t \cos t dt = \frac{e^t \cos t + e^t \sin t}{2} + C.$$

Thus

$$\int \cos(\ln x)dx = \frac{x\cos(\ln x) + x\sin(\ln x)}{2} + C.$$

Problem 5. Evaluate the following integrals.

1. $\int \tan^2\theta \sec^3\theta d\theta$.
Substitution $u = \sec\theta + \tan\theta$ (in lecture), we have

$$du = (\sec\theta\tan\theta + \sec^2\theta)d\theta = u\sec\theta d\theta.$$

Thus

$$\frac{du}{u} = \sec\theta d\theta, \quad \tan\theta = \frac{u^2-1}{u}, \quad \sec\theta = \frac{u^2+1}{u}.$$

Hence

$$\begin{aligned}\int \tan^2\theta\sec^3\theta d\theta &= \int \left(\frac{u^2-1}{2u}\right)^2\left(\frac{u^2+1}{2u}\right)^2\frac{du}{u} \\ &= \frac{1}{16}\int \frac{u^8 - 2u^4 + 1}{u^5}du \\ &= \frac{1}{16}\int (u^3 - \frac{2}{u} + u^{-5})du \\ &= \frac{1}{16}\left(\frac{u^4}{4} - 2\ln|u| - \frac{1}{4u^4}\right) + C\end{aligned}$$

. Substitute $u = \sec\theta + \tan\theta$ we get the answer.

2. $\int \sin^2 t \cos \frac{t}{2} dt$.
First substitute $x = \frac{t}{2}$, $dx = \frac{1}{2}dt$, we have

$$\int \sin^2 t \cos \frac{t}{2} dt = 2 \int \sin^2 2x \cos x dx.$$

Use half angle formula, this equals

$$8 \int \sin^2 x \cos^2 x \cos x dx.$$

Substitution $u = \sin x$, the above integral can be written as

$$\begin{aligned} 8 \int u^2(1-u^2)du &= 8 \int (u^2 - u^4)du \\ &= 8\left(\frac{u^3}{3} - \frac{u^5}{5}\right) + C \\ &= 8\left(\frac{\sin^3(t/2)}{3} - \frac{\sin^5(t/2)}{5}\right) + C. \end{aligned}$$

3. $\int_0^{\pi/4} \frac{\sin^3 x}{\cos x} dx$.
Use substitution with $u = \cos x$, we have

$$\int_0^{\pi/4} \frac{\sin^3 x}{\cos x} dx = -\int_1^{\sqrt{2}/2} \frac{1-u^2}{u} du = -\left(\ln|u| - \frac{u^2}{2}\right)\Bigg|_1^{\sqrt{2}/2} = -\ln(\sqrt{2}/2) + \frac{3}{4}.$$