Math 162 - Spring 2025
Workshop 10
March 7 - March 11
11.8. Power series, 11.9. Representations of functions as power series
11.10. Taylor and Maclaurin series, 11.11. Applications of Taylor polynomials.

Problem 1. Find the radius of convergence and interval of convergence of the following series

a) $\sum_{n=1}^{\infty} \frac{(-1)^n 4^n}{\sqrt{n}} x^n$

b) $\sum_{n=1}^{\infty} \frac{n}{2^n(n^2+1)} x^n$

c) $\sum_{n=1}^{\infty} \frac{(-1)^n}{(2n-1)2^n} (x-1)^n$

d) $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{8^n} (x+6)^n$.

Answer

a) $\sum_{n=1}^{\infty} \frac{(-1)^n 4^n}{\sqrt{n}} x^n$. We apply the Root Test:

$$\lim_{n\to\infty} \left| \frac{(-1)^n 4^n}{\sqrt{n}} x^n \right|^{1/n} = \lim_{n\to\infty} \frac{4|x|}{n^{1/(2n)}} = 4|x|.$$

since $n^{1/(2n)} \to 1$ as $n \to \infty$. The radius of convergence is:

$$R = \frac{1}{4}.$$

b) $\sum_{n=1}^{\infty} \frac{n}{2^n(n^2+1)} x^n$. Apply the Root Test again:

$$\lim_{n\to\infty} \left| \frac{nx^n}{2^n(n^2+1)} \right|^{1/n} = \lim_{n\to\infty} \frac{n^{1/n}|x|}{2(n^2+1)^{1/n}} = \frac{|x|}{2}$$

Because $n^{1/n} \to 1$ and $(n^2+1)^{1/n} \to 1$, we have:

$$R = \frac{1}{\frac{1}{2}} = 2.$$

c) $\sum_{n=1}^{\infty} \frac{(-1)^n}{(2n-1)2^n} (x-1)^n$ Apply the Ratio Test:

$$\left| \frac{a_{n+1}}{a_n} \right| = \left| \frac{(x-1)^{n+1}}{(2(n+1)-1)2^{n+1}} \cdot \frac{(2n-1)2^n}{(x-1)^n} \right| = \frac{(2n-1)}{(2n+1)} \cdot \frac{1}{2}|x-1|$$

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \frac{1}{2}|x-1| \Rightarrow R = \frac{1}{\frac{1}{2}} = 2$$

So the radius of convergence is:

$$R = 2, \quad \text{centered at } a = 1$$

d) $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{8^n}(x+6)^n$. Apply the Ratio Test:

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{\sqrt{n+1}|x+6|^{n+1}}{8^{n+1}} \cdot \frac{8^n}{\sqrt{n}|x+6|^n} = \frac{1}{8} \cdot \sqrt{1+\frac{1}{n}}\,|x+6|$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \frac{1}{8}|x+6| \Rightarrow R = \frac{1}{\frac{1}{8}} = 8.$$

So the radius of convergence is:

$$R = 8, \quad \text{centered at } a = -6.$$

□

Problem 2. Find a power series representation for the function and determine the interval of convergence.

a) $f(x) = \frac{x}{2x^2+1}$.

b) $f(x) = \frac{x}{(1+4x)^2}$.

Answer:

a) $f(x) = \frac{x}{2x^2+1} = x \cdot \frac{1}{1+2x^2}$ Use the geometric series formula:

$$\frac{1}{1-r} = \sum_{n=0}^{\infty} r^n, \quad \text{for } |r| < 1.$$

Let $r = -2x^2$, so:

$$\frac{1}{1+2x^2} = \sum_{n=0}^{\infty}(-2x^2)^n = \sum_{n=0}^{\infty}(-1)^n 2^n x^{2n}$$

Multiplying by x, we get:

$$f(x) = x \cdot \sum_{n=0}^{\infty}(-1)^n 2^n x^{2n} = \sum_{n=0}^{\infty}(-1)^n 2^n x^{2n+1}$$

The series converges when $|r| < 1$, or

$$|-2x^2| < 1 \quad \Rightarrow \quad 2x^2 < 1 \quad \Rightarrow \quad x^2 < \frac{1}{2} \quad \Rightarrow \quad |x| < \frac{1}{\sqrt{2}},$$

So the interval of convergence is:

$$\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right).$$

b) $f(x) = \frac{x}{(1+4x)^2}$. Recall the identity (lecture notes):

$$\frac{1}{(1-r)^2} = \sum_{n=1}^{\infty} n r^{n-1}, \quad \text{for } |r| < 1$$

Let $r = -4x$, then:

$$\frac{1}{(1+4x)^2} = \sum_{n=1}^{\infty} n(-4x)^{n-1}$$

Multiply both sides by x to obtain $f(x)$:

$$f(x) = x \cdot \sum_{n=1}^{\infty} n(-4x)^{n-1} = \sum_{n=1}^{\infty} n(-4)^{n-1} x^n.$$

The series converges when $|r| < 1$ or

$$|-4x| < 1 \quad \Rightarrow \quad |x| < \frac{1}{4}.$$

□

Problem 3. Evaluate the indefinite integrals as power series and determine the radius of convergence:

a) $\int \frac{t}{1+t^3}\, dt$.

b) $\int \frac{\tan^{-1} x}{x}\, dx$.

Answer.

a) $\int \frac{t}{1+t^3}\, dt$. We begin by expressing the integrand as a power series, using geometric series:

$$\frac{t}{1+t^3} = t \cdot \frac{1}{1+t^3} = t \cdot \sum_{n=0}^{\infty} (-1)^n t^{3n} = \sum_{n=0}^{\infty} (-1)^n t^{3n+1}$$

Integrate term by term both sides:

$$\int \frac{t}{1+t^3}\, dt = \sum_{n=0}^{\infty} (-1)^n \frac{t^{3n+2}}{3n+2} + C$$

The geometric series converges when $|t^3| < 1 \Rightarrow |t| < 1$, so the radius of convergence is:

$$R = 1.$$

b) $\int \frac{\tan^{-1} x}{x}\, dx$. We use the known power series expansion:

$$\tan^{-1} x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}, \quad \text{for } |x| < 1$$

Divide by x:

$$\frac{\tan^{-1} x}{x} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2n+1}$$

Now integrate term by term:

$$\int \frac{\tan^{-1} x}{x}\, dx = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \int x^{2n}\, dx = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)^2} + C$$

The radius of convergence is:

$$R = 1.$$

□

Problem 4. Find the Taylor series for $f(x)$ centered at the given value of a. Find the radius of convergence.

a) $f(x) = \ln x$, centered at $a = 2$.

b) $f(x) = x^6 - x^4 + 2, \quad$ centered at $a = -2$.

Answer.

a) Use the Taylor series formula centered at $x = a$:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

Compute the derivatives of $f(x) = \ln x$:

$$f(x) = \ln x, \quad f'(x) = \frac{1}{x}, \quad f''(x) = -\frac{1}{x^2}, \quad f^{(3)}(x) = \frac{2}{x^3}$$
$$f^{(4)}(x) = -\frac{6}{x^4}, \quad \dots \quad , f^{(n)}(x) = (-1)^{n+1}(n-1)! \cdot x^{-n}.$$

Evaluating at $x = 2$:

$$f^{(n)}(2) = (-1)^{n+1}\frac{(n-1)!}{2^n}$$

Substitute into the Taylor series:

$$\ln x = \sum_{n=1}^{\infty} \frac{f^{(n)}(2)}{n!}(x-2)^n = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(n-1)!}{2^n \cdot n!}(x-2)^n = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n \cdot 2^n}(x-2)^n$$

Radius of Convergence. We use the Ratio Test:

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{(x-2)^{n+1}}{(n+1)\cdot 2^{n+1}} \cdot \frac{n \cdot 2^n}{(x-2)^n} \right| = \lim_{n\to\infty} \frac{n}{n+1} \cdot \frac{1}{2}|x-2| = \frac{1}{2}|x-2|.$$

Thus $|x-2| < 2$, and $R = 2$.

b) Recall the Taylor series formula about $x = a$:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = \sum_{n=0}^{\infty} \frac{f^{(n)}(-2)}{n!}(x+2)^n$$

Compute derivatives and evaluate at $x = -2$:

$$f(x) = x^6 - x^4 + 2 \qquad \Rightarrow \qquad f(-2) = 64 - 16 + 2 = 50$$
$$f'(x) = 6x^5 - 4x^3 \qquad \Rightarrow \qquad f'(-2) = -192 + 32 = -160$$
$$f''(x) = 30x^4 - 12x^2 \qquad \Rightarrow \qquad f''(-2) = 480 - 48 = 432$$
$$f^{(3)}(x) = 120x^3 - 24x \qquad \Rightarrow \qquad f^{(3)}(-2) = -960 + 48 = -912$$
$$f^{(4)}(x) = 360x^2 - 24 \qquad \Rightarrow \qquad f^{(4)}(-2) = 1440 - 24 = 1416$$

$$f^{(5)}(x) = 720x \qquad \Rightarrow \qquad f^{(5)}(-2) = -1440$$
$$f^{(6)}(x) = 720 \qquad \Rightarrow \qquad f^{(6)}(-2) = 720$$

All higher-order derivatives are zero since $f(x)$ is a degree 6 polynomial.

Substitute the values into the formula we obtain the Taylor series:

$$f(x) = 50 - 160(x+2) + \frac{432}{2!}(x+2)^2 - \frac{912}{3!}(x+2)^3 + \frac{1416}{4!}(x+2)^4 - \frac{1440}{5!}(x+2)^5 + \frac{720}{6!}(x+2)^6$$

Simplified:

$$f(x) = 50 - 160(x+2) + 216(x+2)^2 - 152(x+2)^3 + 59(x+2)^4 - 12(x+2)^5 + \frac{5}{6}(x+2)^6$$

Since this is a finite-degree polynomial, the Taylor series terminates and:

Radius of convergence: $R = \infty$

□

Problem 5. Use the Maclaurin series from Table 1 to find the series for the following functions:

a) $f(x) = x\cos(2x)$.

b) $f(x) = x\cos\left(\frac{1}{2}x^2\right)$.

c) $f(x) = e^{3x} - e^{2x}$.

d) $f(x) = x^2 \ln(1 + x^3)$.

Answer.

a) We use the Maclaurin series for $\cos(x)$:

$$\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} \Rightarrow \cos(2x) = \sum_{n=0}^{\infty} \frac{(-1)^n (2x)^{2n}}{(2n)!} = \sum_{n=0}^{\infty} \frac{(-1)^n 4^n x^{2n}}{(2n)!}$$

Then multiply by x:

$$f(x) = x\cos(2x) = \sum_{n=0}^{\infty} \frac{(-1)^n 4^n x^{2n+1}}{(2n)!}$$

b)

$$\cos\left(\frac{1}{2}x^2\right) = \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{1}{2}x^2\right)^{2n}}{(2n)!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n}}{2^{2n}(2n)!}$$

Then multiply by x:

$$f(x) = x\cos\left(\frac{1}{2}x^2\right) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+1}}{2^{2n}(2n)!}$$

c) We use the Maclaurin series for e^x:

$$e^{kx} = \sum_{n=0}^{\infty} \frac{(kx)^n}{n!} = \sum_{n=0}^{\infty} \frac{k^n x^n}{n!}$$

Apply it:

$$e^{3x} = \sum_{n=0}^{\infty} \frac{3^n x^n}{n!}, \quad e^{2x} = \sum_{n=0}^{\infty} \frac{2^n x^n}{n!}$$

Therefore:

$$f(x) = e^{3x} - e^{2x} = \sum_{n=0}^{\infty} \frac{(3^n - 2^n)x^n}{n!}.$$

d) We use the Maclaurin series for $\ln(1+x)$:

$$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n}$$

Substitute x^3 for x:

$$\ln(1+x^3) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^{3n}}{n}$$

Then multiply by x^2:

$$f(x) = x^2 \ln(1+x^3) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^{3n+2}}{n}.$$

□

Problem 6. Find the function represented by the given power series:

a) $\sum_{n=0}^{\infty} (-1)^n \frac{x^{4n}}{n!}$.

b) $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^{4n}}{n}$.

Answer:

a) $\sum_{n=0}^{\infty} (-1)^n \frac{x^{4n}}{n!}$. This matches the Maclaurin series for the exponential function:

$$e^{-x} = \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n!}$$

Substituting x^4 for x, we get:

$$e^{-x^4} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n}}{n!}$$

Therefore, the function is:

$$\boxed{f(x) = e^{-x^4}}$$

b) This matches the Maclaurin series for the logarithmic function:

$$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n}$$

Substituting x^4 for x, we get:

$$\ln(1+x^4) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^{4n}}{n}$$

Therefore, the function is:

$$\boxed{f(x) = \ln(1+x^4)}$$

□

Problem 7. Approximate $f(x)$ by a Taylor polynomial of degree n at the number a. Use Taylor's Inequality to estimate the accuracy of the approximation $f(x) \approx T_n(x)$ when x lies in the given interval.

$$f(x) = x^{2/3}, a = 1, n = 3, 0.8 \leq x \leq 1.2$$

Answer: Given $f(x) = x^{2/3}$, $a = 1$, $n = 3$, $0.8 \leq x \leq 1.2$.
First, compute derivatives:

$$f(x) = x^{2/3}, \quad f'(x) = \frac{2}{3}x^{-1/3}, \quad f''(x) = -\frac{2}{9}x^{-4/3}, \quad f^{(3)}(x) = \frac{8}{27}x^{-7/3}$$

Evaluating the values at $x = 1$:

$$f(1) = 1, \quad f'(1) = \frac{2}{3}, \quad f''(1) = -\frac{2}{9}, \quad f^{(3)}(1) = \frac{8}{27}$$

Then the Taylor polynomial of degree 3 is

$$T_3(x) = 1 + \frac{2}{3}(x-1) - \frac{1}{9}(x-1)^2 + \frac{4}{81}(x-1)^3$$

By Taylor's inequality, the remainder is bounded by

$$|R_3(x)| \leq \frac{|f^{(4)}(x)|}{4!}|x-1|^4, \quad \text{for } x \in [0.8, 1.2]$$

Note that $f^{(4)}(x) = -\frac{56}{81}x^{-10/3}$ is increasing so the maximum of $f^{(4)}(x)$ is attained at $x = 1.2$.

$$\Rightarrow |f^{(4)}(x)| \leq \frac{56}{81}(1.2)^{-10/3} \approx 3.76.$$

Also $x \in [0.8, 1.2]$, so $|x-1| \leq 0.2$, hence

$$|R_3(x)| \leq \frac{3.76}{4!}|x-1|^4 = \frac{3.76}{24}0.2^4 = 0.0002510003.$$

□