Math 162 - Spring 2025
Workshop 11 Solution
Apr 14 - Apr 18
11.3.Integral Test, 11.4.Comparison Test ,11.5.Alternating Series

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Problem 1. Use the Integral Test to determine whether the series is convergent or divergent.
(a) $\sum_{n=1}^{\infty} \frac{tan^{-1}(n)}{1+n^2}$. (b) $\sum_{n=2}^{\infty} \frac{1}{nln(n)}$. (c) $\sum_{n=1}^{\infty} \frac{1}{n^2+n^3}$.

Solution:
(a) The function $f(x) = \frac{\tan^{-1} x}{1+x^2}$ is continuous, positive, and decreasing on $(1, \infty)$,
$f'(x) = \frac{(1+x^2)\left(\frac{1}{1+x^2}\right) - (\tan^{-1} x)(2x)}{(1+x^2)^2} = \frac{1-2x\tan^{-1} x}{(1+x^2)^2} < 0\,\text{for}^{2x} \tan^{-1} x > 1$

f$'(x) < 0$ when $x \geq 1 sotheIntegralTestapplies.$

$$\int_1^{\infty} \frac{\tan^{-1} x}{1+x^2}dx = \lim_{t\to\infty} \int_1^t \frac{\tan^{-1} x}{1+x^2}dx = \lim_{t\to\infty} \int_{\tan^{-1} t}^{\tan -1} udu \left[u = \tan^{-1}, x, du = \frac{dx}{1+x^2}\right]$$
$$= \lim_{t\to\infty} \left[\frac{1}{2}u^2\right]_{\tan n^{-1}}^{\tan -1}, = \lim_{t\to\infty} \left[\frac{1}{2}\left(\tan^{-1} t\right)^2 - \frac{1}{2}\left(\tan^{-1} 1\right)^2\right] = \frac{1}{2}\left(\frac{\pi}{2}\right)^2 - \frac{1}{2}\left(\frac{\pi}{4}\right)^2 = \frac{3\pi^2}{32}$$

Since the improper integral is convergent, the serics $\sum_{n=1}^{\infty} \frac{\tan^{-1} n}{1+n^2}$ is also convergent by the Integral Test.

(b) $f(x) = \frac{1}{x\ln x}$ is continuous and positive on $[2, \infty)$, and also decreasing since $f'(x) = -\frac{1+\ln x}{x^2(\ln x)^2} < 0$ for $x > 2$, so we can use the Integral Test. $\int_2^{\infty} \frac{1}{x\ln x}dx = \lim_{t\to\infty}[\ln(\ln x)]_2^t = \lim_{t\to\infty}[\ln(\ln t) - \ln(\ln 2)] = \infty$, so the series $\sum_{n=2}^{\infty} \frac{1}{n\ln n}$ diverges.

(c) The function $f(x) = \frac{1}{x^2+x^3} = \frac{1}{x^2} - \frac{1}{x} + \frac{1}{x+1}$ [by partial fractions] is continuous, positive and decreasing on $[1, \infty)$, so the Integral Test applies.

$$\int_1^{\infty} f(x)dx = \lim_{t\to\infty} \int_1^t \left(\frac{1}{x^2} - \frac{1}{x} + \frac{1}{x+1}\right) dx = \lim_{t\to\infty} \left[-\frac{1}{x} - \ln x + \ln(x+1)\right]_1^t$$
$$= \lim_{t\to\infty} \left[-\frac{1}{t} + \ln\frac{t+1}{t} + 1 - \ln 2\right] = 0 + 0 + 1 - \ln 2$$

The integral converges, so the series $\sum_{n=1}^{\infty} \frac{1}{n^2+n^3}$ converges.

Problem 2. Find the values of p for which the sereies is convergent
(a) $\sum_{n=1}^{\infty} n(1+n^2)^p$. (b) $\sum_{n=1}^{\infty} \frac{ln(n)}{n^p}$.

Solution:
(a) Clearly the series cannot converge if $p \geq -\frac{1}{2}$, because then $\lim_{n\to\infty} n\left(1+n^2\right)^p \neq 0$. So

assume $p < -\frac{1}{2}$. Then $f(x) = x\left(1+x^2\right)^p$ is continuous, positive, and eventually decreasing on $[1, \infty)$, and we can use the Integral Test.

$$\int_1^{\infty} x\left(1+x^2\right)^p dx = \lim_{t\to\infty}\left[\frac{1}{2}\cdot\frac{\left(1+x^2\right)^{p+1}}{p+1}\right]_1^t = \frac{1}{2(p+1)}\lim_{t\to\infty}\left[\left(1+t^2\right)^{p+1} - 2^{p+1}\right]$$

This limit exists and is finite $\Leftrightarrow p+1 < 0 \Leftrightarrow p < -1$, so the series $\sum_{n=1}^{\infty} n\left(1+n^2\right)^p$ converges whenever $p < -1$.

(b) If $p \leq 0, \lim_{n\to\infty}\frac{\ln n}{n^p} = \infty$ and the series diverges, so assume $p > 0. f(x) = \frac{\ln x}{x^p}$ is positive and continuous and $f'(x) < 0$ for $x > e^{1/p}$, so f is eventually decreasing and we can use the Integral Test. Integration by parts gives $\int_1^{\infty}\frac{\ln x}{x^p}dx = \lim_{t\to\infty}\left[\frac{x^{1-p}[(1-p)\ln x - 1]}{(1-p)^2}\right]_1^t$ (for $p \neq 1) = \frac{1}{(1-p)^2}\left[\lim_{t\to\infty} t^{1-p}[(1-p)\ln t - 1] + 1\right]$, which exists whenever $1-p < 0 \Leftrightarrow p > 1$. Thus, $\sum_{n=1}^{\infty}\frac{\ln n}{n^p}$ converges $\Leftrightarrow p > 1$.

Problem 3. Use the Direct Comparison Test to determine whether the series converges or diverges.
(a) $\sum_{k=1}^{\infty}\frac{k\cdot sin^2(k)}{1+k^3}$, (b) $\sum_{n=1}^{\infty}\frac{n!}{n^n}$, (c) $\sum_{n=1}^{\infty}\frac{e^{\frac{1}{n}}}{n}$

Solution:
(a) $\frac{k\sin^2 k}{1+k^3} \leq \frac{k}{1+k^3} < \frac{k}{k^3} = \frac{1}{k^2}$ for all $k \geq 1$, so $\sum_{k=1}^{\infty}\frac{k\sin^2 k}{1+k^3}$ converges by direct comparison with $\sum_{k=1}^{\infty}\frac{1}{k^2}$, which converges because it is a p-series with $p = 2 > 1$.

(b) $\frac{n!}{n^n} = \frac{1\cdot 2\cdot 3\cdots(n-1)n}{n\cdot n\cdot n\cdots n\cdot n} \leq \frac{1}{n}\cdot\frac{2}{n}\cdot 1\cdot 1\cdots 1$ for $n \geq 2$, so since $\sum_{n=1}^{\infty}\frac{2}{n^2}$ converges $[p = 2 > 1]$, $\sum_{n=1}^{\infty}\frac{n!}{n^n}$ converges also by the Direct Comparison Test.

(c) $\frac{e^{1/n}}{n} > \frac{1}{n}$ for all $n \geq 1$, so $\sum_{n=1}^{\infty}\frac{e^{1/n}}{n}$ diverges by direct comparison with the harmonic series $\sum_{n=1}^{\infty}\frac{1}{n}$.

Problem 4. Use the Limit Comparison Test to determine whether the series converges or diverges.
(a) $\sum_{n=1}^{\infty}\sin^2\left(\frac{1}{n}\right)$, (b) $\sum_{n=1}^{\infty}\frac{1}{n^{1+\frac{1}{n}}}$.

Solution:
(a) Use the Limit Comparison Test with $a_n = \sin^2\left(\frac{1}{n}\right)$ and $b_n = \frac{1}{n^2}$:

$$\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\frac{\sin^2\left(\frac{1}{n}\right)}{\frac{1}{n^2}} = \lim_{n\to\infty}\left[\frac{\sin\left(\frac{1}{n}\right)}{\frac{1}{n}}\right]^2 = \lim_{x\to 0}\left(\frac{\sin x}{x}\right)^2 \quad \left[\text{where } x = \frac{1}{n}\right]$$

Now, $\lim_{x\to 0}\frac{\sin x}{x} = 1$ the squaring function is continuous at $x = 1$, so $\lim_{x\to 0}\left(\frac{\sin x}{x}\right)^2 = 1^2 = 1 > 0$. Since $\sum_{n=1}^{\infty}\frac{1}{n^2}$ is a convergent p-series $[p = 2 > 1]$, the series $\sum_{n=1}^{\infty}\sin^2\left(\frac{1}{n}\right)$ also converges.

(b) Use the Limit Comparison Test with $a_n = \frac{1}{n^{1+1/n}}$ and $b_n = \frac{1}{n}$. $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n}{n^{1+1/n}} = \lim_{n\to\infty} \frac{1}{n^{1/n}} = 1$ [since $\lim_{x\to\infty} x^{1/x} = 1$ by l'Hospital's Rule], so $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges [harmonic series] $\Rightarrow \sum_{n=1}^{\infty} \frac{1}{n^{1+1/n}}$ diverges.

Problem 5. Test the series for convergence or divergence.
(a) $\sum_{n=1}^{\infty} \frac{(-1)^n n^2}{5^n}$, (b) $\sum_{n=1}^{\infty} (-1)^n \left(\sqrt{n+1} - \sqrt{n}\right)$, (c) $\sum_{n=1}^{\infty} \frac{n\cos(n\pi)}{2^n}$.

Solution:
(a) $b_n = \frac{n^2}{5^n} > 0$ for $n \geq 1$. $\{b_n\}$ is decreasing for $n \geq 2$ since

$$\left(\frac{x^2}{5^x}\right)' = \frac{5^x \cdot 2x - x^2 5^x \ln 5}{(5^x)^2} = \frac{x5^x(2 - x\ln 5)}{(5^x)^2} = \frac{x(2 - x\ln 5)}{5^x} < 0 \text{ for } x > \frac{2}{\ln 5} \approx 1.2. \text{ Also,}$$

$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{n^2}{5^n} \overset{H}{=} \lim_{n\to\infty} \frac{2n}{5^n \ln 5} \overset{H}{=} \lim_{n\to\infty} \frac{2}{5^n (\ln 5)^2} = 0$. Thus, the series $\sum_{n=1}^{\infty} (-1)^n \frac{n^2}{5^n}$ converges by the Alternating.

(b)$b_n = \frac{\sqrt{n+1} - \sqrt{n}}{1} \cdot \frac{\sqrt{n+1} + \sqrt{n}}{\sqrt{n+1} + \sqrt{n}} = \frac{(n+1) - n}{\sqrt{n+1} + \sqrt{n}} = \frac{1}{\sqrt{n+1} + \sqrt{n}} > 0$ for $n \geq 1$. $\{b_n\}$ is decreasing and $\lim_{n\to\infty} b_n = 0$, so the series $\sum_{n=1}^{\infty} (-1)^n (\sqrt{n+1} - \sqrt{n})$ converges by the Alternating Series Test.

(c) $a_n = \frac{n\cos n\pi}{2^n} = (-1)^n \frac{n}{2^n} = (-1)^n b_n$. $\{b_n\}$ is decreasing for $n \geq 2$ since $(x2^{-x})' = x\left(-2^{-x}\ln 2\right) + 2^{-x} = 2^{-x}(1 - x\ln 2) < 0$ for $x > \frac{1}{\ln 2}[\approx 1.4]$. Also, $\lim_{n\to\infty} b_n = 0$ since $\lim_{x\to\infty} \frac{x}{2^x} \overset{H}{=} \lim_{x\to\infty} \frac{1}{2^x \ln 2} = 0$. Thus, the series $\sum_{n=1}^{\infty} \frac{n\cos n\pi}{2^n}$ converges by the Alternating Series Test.

Problem 6. Determine whether the series is absolutely convergent, conditionally convergent, or divergent
(a) $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^{\frac{2}{3}}}$, (b) $\sum_{n=1}^{\infty} \frac{1 + 2\sin(n)}{n^3}$, (c) $\sum_{n=0}^{\infty} (-1)^{n+1} \frac{n^2}{n^2+1}$,

Solution:
(a) $b_n = \frac{1}{\sqrt[3]{n^2}} > 0$ for $n \geq 1$. $\{b_n\}$ is decreasing for $n \geq 1$, and $\lim_{n\to\infty} \frac{1}{\sqrt[3]{n^2}} = 0$, so the series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\sqrt[3]{n^2}}$ converges by the Alternating Series Test. Also, observe that $\sum_{n=1}^{\infty} \left|\frac{(-1)^{n-1}}{\sqrt[3]{n^2}}\right| = \sum_{n=1}^{\infty} \frac{1}{n^{2/3}}$ is divergent since it is a p-series with $p = \frac{2}{3} \leq 1$. Thus, the series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\sqrt[3]{n^2}}$ is conditionally convergent.

(b) $0 < \left|\frac{1+2\sin n}{n^3}\right| < \frac{3}{n^3}$ for $n \geq 1$ and $3\sum_{n=1}^{\infty} \frac{1}{n^3}$ is a constant times a convergent p-series [$p = 3 > 1$], so $\sum_{n=1}^{\infty} \left|\frac{1+2\sin n}{n^3}\right|$ converges by direct comparison and the series $\sum_{n=1}^{\infty} \frac{1+2\sin n}{n^3}$ is absolutely convergent.

(c) Since $\lim_{n\to\infty} \frac{n^2}{n^2+1} = \lim_{n\to\infty} \frac{1}{1+1/n^2} = \frac{1}{1+0} = 1 \neq 0$ and $\lim_{n\to\infty}(-1)^{n+1}$ does not exist, $\lim_{n\to\infty}(-1)^{n+1}\frac{n^2}{n^2+1}$ does not exist, so the series $\sum_{n=0}^{\infty}(-1)^{n+1}\frac{n^2}{n^2+1}$ diverges by the Test for Divergence.