Math 162 - Spring 2025
Workshop 10
March 7 - March 11
11.1.More on sequences, 11.2 Series.

Problem 1. Consider the sequence defined by:

$$a_1 = 1, \quad a_{n+1} = 3 - \frac{1}{a_n}.$$

- Show that the sequence is increasing.
- Show that the sequence is bounded above by 3.
- Deduce that $\{a_n\}$ is convergent and find its limit.

Answer

Step 1: Show that $1 \leq a_n < 3$ for all n.
We proceed by induction. *Base case:* $1 \leq a_1 = 1 < 3$.
Inductive step: Assume $1 \leq a_n < 3$. Then

$$a_{n+1} = 3 - \frac{1}{a_n} < 3$$

since $a_n \geq 1$, and

$$a_{n+1} = 3 - \frac{1}{a_n} \geq 1$$

since $a_n < 3$, completing the induction.

Step 2: Show that the sequence is increasing.
We can do so using induction again, with the base case being $a_1 = 1 \leq a_2 = 2$.
Now we assume $a_n \leq a_{n+1}$ and want to show $a_{n+1} \leq a_{n+2}$. So, we have

$$a_n \leq a_{n+1} \implies \frac{-1}{a_n} \leq \frac{-1}{a_{n+1}} \implies 3 - \frac{1}{a_n} \leq 3 - \frac{1}{a_{n+1}} \implies a_{n+1} \leq a_{n+2}.$$

Step 3: Conclude convergence
The sequence $\{a_n\}$ is increasing and bounded above by 3. By the Monotone Convergence Theorem, it must converge.

Step 4: Find the limit
Let $\lim_{n\to\infty} a_n = L$. Then taking limits in the recurrence:

$$L = 3 - \frac{1}{L} \Rightarrow L^2 = 3L - 1 \Rightarrow L^2 - 3L + 1 = 0.$$

Solving the quadratic:

$$L = \frac{3 \pm \sqrt{9-4}}{2} = \frac{3 \pm \sqrt{5}}{2}.$$

The limit must be greater than $a_1 = 1$. So we have

$$\lim_{n\to\infty} a_n = \boxed{\frac{3+\sqrt{5}}{2}}.$$

□

Problem 2. Determine whether the sequence converges or diverges. If it converges, find the limit.

a) $a_n = \frac{4^n}{1+9^n}$.

b) $a_n = \cos\left(\frac{n\pi}{n+1}\right)$.

Answer:

a) $a_n = \frac{4^n}{1+9^n} \leq \left(\frac{4}{9}\right)^n$, by Squeeze Theorem, $a_n \to 0$.

b) $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \cos\left(\frac{n\pi}{n+1}\right) = \cos\left(\lim_{n\to\infty} \frac{n\pi}{n}\right) = \cos(\pi) = -1$.

□

Problem 3. Determine whether the series is convergent or divergent by expressing s_n as a telescoping sum. If it is convergent, find its sum.

a) $\sum_{n=1}^{\infty} \left(\frac{1}{n+2} - \frac{1}{n}\right)$.

b) $\sum_{n=4}^{\infty} \left(\frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n+1}}\right)$.

Answer.

a) The partial sum

$$\begin{aligned} s_n &= \sum_{k=1}^{n} \left(\frac{1}{k+2} - \frac{1}{k}\right) \\ &= \left(\frac{1}{3} - \frac{1}{1}\right) + \left(\frac{1}{4} - \frac{1}{2}\right) + \left(\frac{1}{5} - \frac{1}{3}\right) + \left(\frac{1}{6} - \frac{1}{4}\right) \\ &\qquad + \cdots + \left(\frac{1}{n} - \frac{1}{n-2}\right) + \left(\frac{1}{n+1} - \frac{1}{n-1}\right) + \left(\frac{1}{n+2} - \frac{1}{n}\right) \\ &= -1 - \frac{1}{2} + \frac{1}{n+1} + \frac{1}{n+2}. \end{aligned}$$

Thus $\sum a_n = \lim_{n\to\infty} s_n = -\frac{3}{2}$.

b) The partial sum is

$$s_n = \sum_{n=4}^{n} \left(\frac{1}{\sqrt{k}} - \frac{1}{\sqrt{k+1}}\right) = \frac{1}{2} - \frac{1}{\sqrt{n+1}}.$$

Thus $\sum a_n = \lim_{n\to\infty} s_n = \frac{1}{2}$.

□

Problem 4. Determine whether the geometric series is convergent or divergent. If it is convergent, find its sum.

a) $\sum_{n=1}^{\infty} \frac{e^{2n}}{6^{n-1}}$.

b) $\sum_{n=1}^{\infty} \frac{6 \cdot 2^{2n-1}}{3^n}$.

c) $2 + 0.5 + 0.125 + 0.03125 + \cdots$

Answer.

a) Write

$$\frac{e^{2n}}{6^{n-1}} = e^2 \cdot \left(\frac{e^2}{6}\right)^{n-1}.$$

Thus

$$\sum a_n = \sum_{n=1}^{\infty} e^2 \left(\frac{e^2}{6}\right)^{n-1}.$$

Note that $r = \frac{e^2}{6} \approx \frac{7.389}{6} > 1$, so the common ratio is greater than 1, and the series diverges.

b) Write the series as follows

$$\sum_{n=1}^{\infty} \frac{6 \cdot \frac{4^n}{2}}{3^n} = \sum_{n=1}^{\infty} \frac{3 \cdot 4^n}{3^n} = \sum_{n=1}^{\infty} 3 \left(\frac{4}{3}\right)^n.$$

This is a geometric series with $r = \frac{4}{3} > 1$, so it diverges.

c) Consider the series:

$$2 + 0.5 + 0.125 + 0.03125 + \cdots$$

We observe that this is a geometric series where each term is obtained by multiplying the previous term by 0.25. Thus, it is a geometric series with:

$$a = 2, \quad r = 0.25.$$

Since $|r| < 1$, the series converges. The sum of an infinite geometric series is given by:

$$S = \frac{a}{1-r}.$$

Substituting the values:

$$S = \frac{2}{1-0.25} = \frac{2}{0.75} = \frac{8}{3}.$$

□

Problem 5. Determine whether the series is convergent or divergent. If it is convergent, find its sum.

a) $\sum_{n=1}^{\infty} \left(\frac{2+n}{1-2n}\right)$.

b) $\sum_{k=1}^{\infty} \frac{k^2}{k^2-2k+5}$.

c) $\sum_{n=1}^{\infty} \left(\frac{3}{5^n} + \frac{2}{n}\right)$.

d) $\sum_{n=1}^{\infty} \left(\frac{1}{e^n} + \frac{1}{n(n+1)}\right)$.

Answer.

a)

$$\lim_{n\to\infty} \frac{2+n}{1-2n} = -\frac{1}{2}.$$

Thus the series diverges by the Divergence Test.

b) As $k \to \infty$:

$$\frac{k^2}{k^2 - 2k + 5} \to 1$$

Diverges.

c) Split into two series:

$$\sum_{n=1}^{\infty} \left(\frac{3}{5^n} + \frac{2}{n}\right) = \sum_{n=1}^{\infty} \frac{3}{5^n} \quad \text{(geometric, converges)} + \sum_{n=1}^{\infty} \frac{2}{n} \quad \text{(harmonic, diverges)}.$$

Since one component diverges, the full series diverges.

d) Split into two series:

$$\sum_{n=1}^{\infty} \frac{1}{e^n} = \frac{1}{e-1} \quad \text{(geometric)}$$

Use partial fractions:

$$\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1} \Rightarrow \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1}\right) = 1$$

Total sum:

$$\boxed{1 + \frac{1}{e-1}}$$

□