Math 162: Calculus IIA

Second Midterm Exam ANSWERS

November 6, 2019

1. (20 points)

Find the arc length of the curve described by the parametric equations

$$x = 1 + 3t^2, \quad y = 4 + 2t^3$$

between the points with Cartesian coordinates $(1, 4)$ and $(4, 6)$.

Answer:

The points on the curve with Cartesian coordinates $(1, 4)$ and $(4, 6)$ are the points when the parameter t equals 0 and 1 respectively.

We have that

$$\begin{aligned} dx/dt = 6t, \quad dy/dt &= 6t^2 \\ (dx/dt)^2 = 36t^2, \quad (dy/dt)^2 &= 36t^4 \\ (dx/dt)^2 + (dy/dt)^2 &= 36t^2(1+t^2) \\ \sqrt{(dx/dt)^2 + (dy/dt)^2} &= 6t\sqrt{1+t^2} \end{aligned}$$

So the arc length L is

$$L = \int_0^1 \sqrt{(dx/dt)^2 + (dy/dt)^2} dt = \int_0^1 6t\sqrt{1+t^2} dt$$

.

Make the substitution $u = 1 + t^2$. Then $du = 2tdt$, and when $t = 0$ we have that $u = 1$, and when $t = 1$, $u = 2$. So

$$L = 3\int_1^2 u^{1/2} du = 3\Big[2/3u^{3/2}\Big]_1^2 = 2(2^{3/2} - 1) = 2(2\sqrt{2} - 1) = 4\sqrt{2} - 2.$$

2. (20 points) Determine if the following sequences are convergent or divergent and explain why. If it is convergent, give its limit.

(a) $\left\{ \frac{n\cos(n)}{n^2+1} \,\middle|\, n \geq 0 \right\}$.

Answer:

Since for every $n \geq 1$ we have $-1 \leq \cos(n) \leq 1$ and hence

$$\frac{-n}{n^2+1} \leq \frac{n\cos(n)}{n^2+1} \leq \frac{n}{n^2+1}.$$

Consider

$$\lim_{n\to\infty} \frac{-n}{n^2+1} = \lim_{n\to\infty} \frac{n}{n^2+1} = 0.$$

By Squeeze Theorem, we prove $\lim\limits_{n\to\infty} \frac{n\cos(n)}{n^2+1} = 0$, and hence the sequence is convergent.

(b) $\left\{ n^3 \sin\left(\frac{1}{n}\right) \,\middle|\, n \geq 1 \right\}$

Answer:

Putting $x = 1/n$ we have

$$\lim_{n\to\infty} n^3 \sin\left(\frac{1}{n}\right) = \lim_{x\to 0^+} \frac{\sin x}{x^3} = \lim_{x\to 0^+} \frac{\cos x}{3x^2} = \infty.$$

Therefore, it is divergent.

3. (20 points) Compute the following integral:

$$\int \frac{1}{(x^2-1)^2} dx.$$

Answer:

$\frac{1}{(x^2-1)^2} = \frac{1}{(x-1)^2(x+1)^2}$. The partial fraction decomposition can be written as

$$\frac{1}{(x-1)^2(x+1)^2} = \frac{A}{(x-1)} + \frac{B}{(x+1)} + \frac{C}{(x-1)^2} + \frac{D}{(x+1)^2}.$$

Multiplying the above equation by $(x-1)^2(x+1)^2$ we get

$$1 = A(x-1)(x+1)^2 + B(x-1)^2(x+1) + C(x+1)^2 + D(x-1)^2. \tag{1}$$

Substituting $x = 1$ into (1) we get $1 = C \cdot 4$, implying that $C = 1/4$.

Substituting $x = -1$ into (1) we get $1 = D \cdot 4$, implying that $D = 1/4$.

Now, let us expand each term in (1) and sort by degree:

$$1 = A(x^3 + x^2 - x - 1) + B(x^3 - x^2 - x + 1) + \frac{1}{4}(x^2 - 2x + 1) + \frac{1}{4}(x^2 + 2x + 1)$$
$$1 = (A+B)x^3 + (A - B + \frac{1}{2})x^2 + (-A-B)x + (-A + B + \frac{1}{2}).$$

Consider the coefficients of the x^3 and x^2 terms. On the left hand side, the coefficients of x^3 and x^2 are 0, so the coefficients on the right hand side should equal 0. This gives the following equations:

$$A + B = 0, A - B + \frac{1}{2} = 0.$$

Solving these equations gives

$$A = -\frac{1}{4}, B = \frac{1}{4}.$$

Therefore,

$$\begin{aligned}
\int \frac{1}{(x^2-1)^2} dx &= \int \left(\frac{-1}{4(x-1)} + \frac{1}{4(x+1)} + \frac{1}{4(x-1)^2} + \frac{1}{4(x+1)^2} \right) dx \\
&= \frac{1}{4} \left(\int \frac{-dx}{(x-1)} + \int \frac{dx}{(x+1)} + \int \frac{dx}{(x-1)^2} + \int \frac{dx}{(x+1)^2} \right) \\
&= \frac{1}{4} \left(-\ln|x-1| + \ln|x+1| - \frac{1}{x-1} - \frac{1}{x+1} \right) + C.
\end{aligned}$$

4. (20 points)

(a) Compute the area of surface of revolution obtained by rotating the curve $y = \sqrt{4 - x^2}$ around the x-axis.

Answer:

$$A \quad = \quad 2\pi \int_{-2}^{2} y \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, dx$$

$$= 2\pi \int_{-2}^{2} \sqrt{4-x^2}\sqrt{1+\frac{x^2}{4-x^2}}\,dx$$
$$= 2\pi \int_{-2}^{2} 2\,dx = 16\pi$$

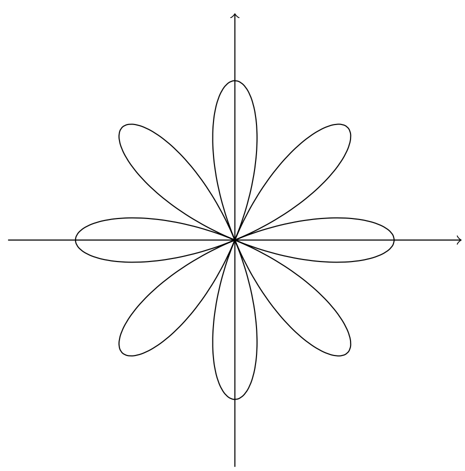
(b) Do the same for the curve $y = 1 - |x|$, $-1 \le x \le 1$.

Answer:

$$\begin{aligned} A &= 2\pi \int_{-1}^{0} (1+x)\sqrt{1+1}\,dx + 2\pi \int_{0}^{1} (1-x)\sqrt{1+1}\,dx \\ &= 2\pi\sqrt{2}\left(\left[x+\frac{x^2}{2}\right]_{-1}^{0} + \left[x-\frac{x^2}{2}\right]_{0}^{1}\right) \\ &= 2\pi\sqrt{2}\left(\frac{1}{2}+\frac{1}{2}\right) = 2\pi\sqrt{2} \end{aligned}$$

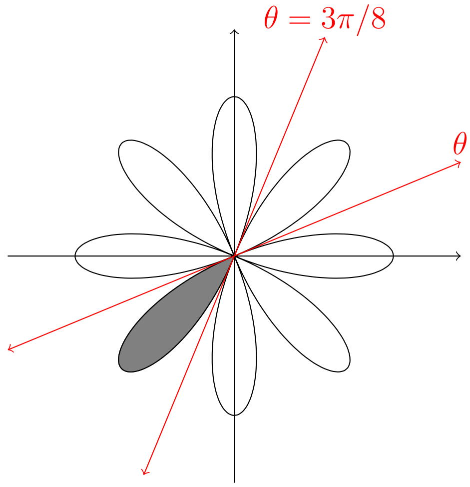
5. (20 points)

(a) Find the area of **one petal** of the polar rose $r = 2\cos(4\theta)$ pictured below.



Answer:

We need to find consecutive zeros of $r = 2\cos(4\theta)$. These will give the limits of integration, because the petal will close for those θ values. If $0 = 2\cos(4\theta)$, then $4\theta = \pi/2, 3\pi/2$, so $\theta = \pi/8, 3\pi/8$ are the limits of integration.



$$\begin{aligned} \text{Area} &= \frac{1}{2}\int_{\pi/8}^{3\pi/8} 4\cos^2(4\theta)d\theta \\ &= 2\int_{\pi/8}^{3\pi/8} \frac{1+\cos(8\theta)}{2}d\theta \\ &= \theta + \frac{\sin(8\theta)}{8}\Big|_{\pi/8}^{3\pi/8} \\ &= \pi/4 \end{aligned}$$

Other correct integrals: $\frac{1}{8\frac{1}{2}}\int_0^{2\pi} 4\cos^2(4\theta)d\theta$, $2\frac{1}{2}\int_0^{\pi/8} 4\cos^2(4\theta)d\theta$, $\frac{1}{2}\int_{3\pi/8}^{5\pi/8} 4\cos^2(4\theta)d\theta$, etc.

(b) The parametric curve given by $x = 4t^3 - 3t$, $y = t^2 + 1$ intersects the y-axis at 3 different values of t. What are the **equations of the tangent lines** to the curve at each of these points?

Answer:

Solve $x = 4t^3 - 3t = 0$ to get $t = 0, \pm\frac{\sqrt{3}}{2}$. We have

$$\frac{dy}{dx} = \frac{2t}{12t^2 - 3}$$

At $t = 0$, the tangent is horizontal with y-intercept $y = 1$, so we get $y = 1$. At $t = \sqrt{3}/2$, at $(x, y) = (0, 1 + 3/4)$, the tangent has slope $\sqrt{3}/6$. At $t = -\sqrt{3}/2$, also at $(x, y) = (0, 1 + 3/4)$, the tangent has slope $-\sqrt{3}/6$. So the two lines are

$$y = \pm\frac{\sqrt{3}}{6}x + \frac{7}{4}.$$

Scratch paper

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