Math 162: Calculus IIA

First Midterm Exam ANSWERS

October 8, 2019

1. (20 points)

Evaluate the indefinite integral:

$$\int \tan^3(Ax+B)dx$$

Answer:

$\tan^3(Ax+B) = (\sec^2(Ax+B)-1)\tan(Ax+b)$

$$\int \tan^3(Ax+B)dx = \int \tan(Ax+B)\sec^2(Ax+B)dx - \int \tan(Ax+B)dx)$$

Let $v = Ax+B$. Then $dv = Adx$, $\int \tan(Ax+B)dx = (1/A)\int \tan v dv = (1/A)\ln|\sec v| = (1/A)\ln|\sec(Ax+B)|$

$u = \tan(Ax+B))$

$du = A\sec^2(Ax+B)dx$

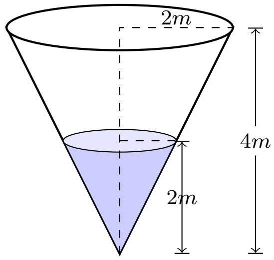
$$\int \tan(Ax+B)\sec^2(Ax+B)dx = (1/A)\int udu = (u^2/2A) = (1/2A)\tan^2(Ax+B)$$

So

$$\int \tan^3(Ax+B)dx = \int \tan(Ax+B)\sec^2(Ax+B)dx - \int \tan(Ax+B)dx)$$

$$= (1/2A)\tan^2(Ax+B) + (1/A)\ln|sec(Ax+B)| + C$$

2. (20 points) A cone shaped tank 4 meters high with a radius of 2 meters at the top contains water of height 2 meters. Find the work done pumping the water to the top of the tank. (The water density is $\rho = 1000kg/m^3$ and the gravity constant is $g = 10m/s^2$)

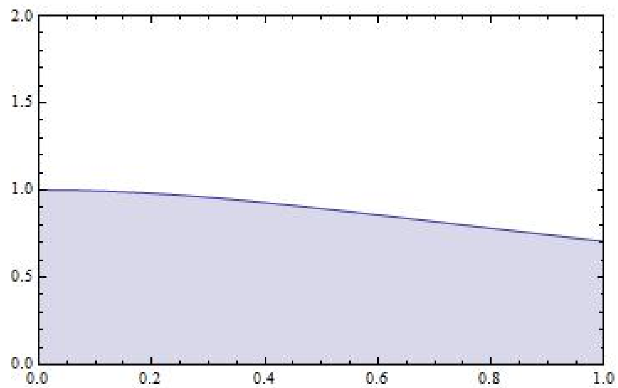


Answer:

Put the bottom of this tank as the origin. For the disk of height y, by relation of similar triangles, we have that its radius $r(y) = \frac{2y}{4}$. Therefore, we have

$$W = \int_0^2 r(y)^2 \pi\rho g(4-y)dy = \frac{\pi\rho g}{4}\int_0^2 4y^2 - y^3 dy = \frac{\pi\rho g}{4}(\frac{4y^3}{3} - \frac{y^4}{4})\Big]_0^2 = \frac{5}{3}\pi\rho g J.$$

3. (20 points)



Consider the region $\mathcal{R}$ bounded by the x-axis, y-axis, the line $x = 1$, and $y = \frac{1}{\sqrt{1+x^2}}$.

(a) **(10 points)** Compute the volume of the solid obtained by revolving $\mathcal{R}$ about the x-axis.

Answer:

We can use the disk method. The volume equals

$$\begin{aligned}\int_0^1 \pi\left(\frac{1}{\sqrt{1+x^2}}\right)^2 dx =& \pi\int_0^1 \frac{1}{1+x^2}dx \\ =& \pi\left[\arctan(x)\right]_0^1 \\ =& \frac{\pi^2}{4}\end{aligned}$$

(b) **(10 points)** Compute the volume of the solid obtained by revolving $\mathcal{R}$ about the y-axis.

Answer:

We can use the shell method. The volume equals

$$\int_0^1 2\pi x \frac{1}{\sqrt{1+x^2}} dx = \pi \int_0^1 \frac{2x}{\sqrt{1+x^2}} dx$$

Now, we can let $u = 1 + x^2$ and $f(u) = \frac{1}{\sqrt{u}}$. So $du = 2xdx$.

$$\begin{aligned} \pi \int \frac{2x}{\sqrt{1+x^2}} dx =& \pi \int \frac{du}{\sqrt{u}} \\ =& 2\pi\sqrt{u} + C \\ =& 2\pi\sqrt{1+x^2} + C. \end{aligned}$$

So the definite integral is

$$\begin{aligned} \pi \int_0^1 \frac{2x}{\sqrt{1+x^2}} dx =& 2\pi[\sqrt{1+x^2}]_0^1 \\ =& 2\pi\left(\sqrt{2} - 1\right). \end{aligned}$$

4. (20 points)

(a) (10 points) Use integration by parts to find a formula for

$$\int x^{2n} \sin x \, dx \qquad \text{in terms of} \qquad \int x^{2n-2} \sin x \, dx$$

Answer:

Using integration by parts twice we have

$$\begin{aligned} \int x^{2n} \sin x \, dx &= \int x^{2n} (-\cos x)' \, dx \\ &= -x^{2n} \cos x + 2n \int x^{2n-1} \cos x \, dx \\ &= -x^{2n} \cos x + 2n \int x^{2n-1} (\sin x)' \, dx \\ &= -x^{2n} \cos x + 2nx^{2n-1} \sin x - 2n(2n-1) \int x^{2n-2} \sin x \, dx \end{aligned}$$

(b) (10 points) Use this formula to find

$$\int x^4 \sin x \, dx.$$

Answer:

$$\begin{aligned}
\int x^4 \sin x \, dx &= -x^4 \cos x + 4x^3 \sin x - 4 \cdot 3 \int x^2 \sin x \, dx \\
&= -x^4 \cos x + 4x^3 \sin x - 12 \left(-x^2 \cos x + 2x \sin x - 2 \int \sin x \, dx \right) \\
&= -x^4 \cos x + 4x^3 \sin x - 12 \left(-x^2 \cos x + 2x \sin x + 2 \cos x \right) + C \\
&= (-x^4 + 12x^2 - 24) \cos x + (4x^3 - 24x) \sin x + C
\end{aligned}$$

5. (20 points) (a) (10 points) Find the integral

$$\int_{-1}^{0} \frac{dx}{\sqrt{x^2 + 4x + 3}}$$

Answer:

We have

$$x^2 + 4x + 3 = (x + 2)^2 - 1$$

We use the substitution $x + 2 = \sec \theta$. Then we have

$$\begin{aligned}
dx &= \sec \theta \tan \theta d\theta \\
\sqrt{x^2 + 4x + 3} &= \tan \theta
\end{aligned}$$

so our integral is

$$\begin{aligned}
\int_{-1}^{0} \frac{dx}{\sqrt{x^2 + 4x + 3}} &= \int_{0}^{\pi/3} \sec \theta d\theta \\
&= \ln |\sec \theta + \tan \theta|_{0}^{\pi/3} \\
&= \ln(\sqrt{3} + 2)
\end{aligned}$$

(b) (10 points) Find the integral

$$\int_{4}^{6} \sqrt{8x - x^2} dx.$$

Answer:

We have

$$8x - x^2 = 16 - (x-4)^2$$

We use the substitution $x - 4 = 4\sin\theta$. From this we get

$$\begin{aligned}
dx &= 4\cos\theta d\theta \\
\sqrt{8x - x^2} &= 4\cos\theta \\
\int_0^{\pi/6} (4\cos\theta) 4\cos\theta d\theta &= 16\int_0^{\pi/6} \cos^2\theta d\theta \\
&= 8\int_0^{\pi/6} (1 + \cos 2\theta) d\theta \\
&= 8\left(\theta + \frac{\sin 2\theta}{2}\right)\Big|_0^{\pi/6} \\
&= \frac{4}{3}\pi + 2\sqrt{3}.
\end{aligned}$$

Scratch paper

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