Math 162: Calculus IIA

First Midterm Exam ANSWERS

October 16, 2014

1. (20 points)

(a) Use integration by parts to express $\int x^n e^x\,dx$ in terms of $\int x^{n-1}e^x\,dx$ for $n > 0$.

(b) Use the formula repeatedly to find

$$\int x^3 e^x\,dx.$$

You will not get partial credit here if the formula you are using is incorrect.

Solution: a.) Let $u = x^n$ and $dv = e^x\,dx$, so $du = nx^{n-1}\,dx$ and $v = e^x$. Then

$$\begin{aligned}\int x^n e^x dx = \int u\,dv &= uv - \int v\,du \\ &= x^n e^x - n\int x^{n-1}e^x\,dx\end{aligned}$$

b.) We have

$$\begin{aligned}\int x^3 e^x dx &= x^3e^x - 3\int x^2e^x\,dx \\ &= x^3e^x - 3\left(x^2e^x - 2\int xe^x\,dx\right) \\ &= x^3e^x - 3\left(x^2e^x - 2\left(xe^x - \int e^x\,dx\right)\right) \\ &= x^3e^x - 3\left(x^2e^x - 2\left(xe^x - e^x\right)\right) + c \\ &= \left(x^3 - 3x^2 + 6x - 6\right)e^x + c.\end{aligned}$$

2. (20 points) The half bagel problem. Consider the region between the x-axis and the semicircle $y = \sqrt{a^2 - (x-b)^2}$ with $b > a > 0$. The semicircle has radius a and center $(b, 0)$. We want to find the volume V of the solid of revolution about the y-axis.

(a) Write the integral for the volume and convert it to a trig integral using the substitution $x = b - a\cos\theta$ for $0 \le \theta \le \pi$.

(b) Find the volume in terms of a and b.

You will not get partial credit here if the integral you are using is incorrect.

Solution: (a) Since $x = b - a\cos\theta$, we have $dx = a\sin\theta\, d\theta$ and

$$y = \sqrt{a^2 - (x-b)^2} = \sqrt{a^2 - (-a\cos\theta)^2} = a\sqrt{1-\cos^2\theta} = a\sin\theta.$$

The integral for the volume is

$$V = \int_{b-a}^{b+a} 2\pi xy\, dx = 2\pi \int_0^{\pi} (b - a\cos\theta)(a\sin\theta)a\sin\theta\, d\theta$$

(b) Our integral is

$$\begin{aligned}
V &= 2\pi \int_0^{\pi} (b - a\cos\theta)(a\sin\theta)a\sin\theta\, d\theta \\
&= 2\pi a^2 b \int_0^{\pi} \sin^2\theta\, d\theta - 2\pi a^3 \int_0^{\pi} \sin^2\theta\cos\theta\, d\theta \\
&= 2\pi a^2 b \int_0^{\pi} \frac{1-\cos 2\theta}{2}\, d\theta - 2\pi a^3 \int_0^{0} u^2\, du \qquad \text{where } u = \sin\theta \\
&= 2\pi a^2 b \int_0^{2\pi} \frac{1-\cos w}{4}\, dw \qquad \text{where } w = 2\theta \\
&= \frac{\pi a^2 b}{2} \left. (w - \sin w)\right|_0^{2\pi}\, dw \\
&= \pi^2 a^2 b.
\end{aligned}$$

3. (20 points)

(a) Let $a > 0$ be a fixed positive number. Compute the definite integral

$$\int_0^{a/\sqrt{2}} \frac{1}{(a^2 - x^2)^{3/2}}\, dx.$$

Your answer should be expressed in terms of a.

(b) Find the integral

$$\int \frac{1}{\sqrt{x^2 + 2x + 10}}\, dx.$$

Solution: (a)

We set $x = a\sin\theta$. Then $dx = a\cos\theta d\theta$ and $\sqrt{a^2 - x^2} = a\cos\theta$. Also when $x = 0$, $\sin\theta = 0$ so that $\theta = 0$, and when $x = \frac{a}{\sqrt{2}}$, $\sin\theta = \frac{1}{\sqrt{2}}$ so that $\theta = \frac{\pi}{4}$. The definite integral becomes

$$\int_0^{a/\sqrt{2}} \frac{1}{(a^2 - x^2)^{3/2}}\,dx = \int_0^{\pi/4} \frac{a\cos\theta}{(a\cos\theta)^3}\,d\theta$$
$$= \frac{1}{a^2}\int_0^{\pi/4} \sec^2\theta\,d\theta = \frac{1}{a^2}\tan\theta|_0^{\pi/4} = \frac{1}{a^2}.$$

(b) We complete the square $x^2 + 2x + 10 = (x+1)^2 + 9$. Then consider the substitution $u = x + 1$, so that $du = dx$, and we find

$$\int \frac{1}{\sqrt{x^2 + 2x + 10}}\,dx = \int \frac{1}{\sqrt{(x+1)^2 + 9}}\,dx = \int \frac{1}{\sqrt{u^2 + 9}}\,du.$$

Next we use a trig substitution. Let $u = 3\tan\theta$. Then $du = 3\sec^2\theta d\theta$ and $\sqrt{u^2 + 9} = 3\sec\theta$, so that

$$\int \frac{1}{\sqrt{x^2 + 2x + 10}}\,dx = \int \frac{1}{\sqrt{u^2+9}}\,du = \int \frac{1}{3\sec\theta} 3\sec^2\theta\,d\theta = \int \sec\theta\,d\theta$$
$$= \ln|\sec\theta + \tan\theta| + C = \ln|\frac{\sqrt{u^2+9}}{3} + \frac{u}{3}| + C$$
$$= \ln|\frac{\sqrt{(x+1)^2+9}}{3} + \frac{x+1}{3}| + C$$
$$= \ln|\sqrt{(x+1)^2+9} + x + 1| + C$$

4. (20 points)

(a) Evaluate the integral

$$\int \frac{2x^3 + 5x^2 + x}{x^3 + x^2 - x - 1}\,dx.$$

(b) Find the integral

$$\int \sin^5 x \cos^2 x\,dx.$$

Solution: (a)

Note that $2x^3 + 5x^2 + x = (x^3 + x^2 - x - 1)\cdot 2 + (3x^2 + 3x + 2)$. So,

$$\frac{2x^3 + 5x^2 + x}{x^3 + x^2 - x - 1} = 2 + \frac{3x^2 + 3x + 2}{x^3 + x^2 - x - 1}.$$

Factoring the denominator,

$$x^3 + x^2 - x - 1 = (x+1)^2(x-1).$$

So, we may write

$$\frac{3x^2+3x+2}{x^3+x^2-x-1} = \frac{A}{x-1} + \frac{B}{x+1} + \frac{C}{(x+1)^2}.$$

Summing the terms on the right hand side and comparing the numerators on both sides, we get

$$3x^2 + 3x + 2 = A(x+1)^2 + B(x-1)(x+1) + C(x-1).$$

Plugging $x = 1$ in the above equation, $8 = 4A$. So, $A = 2$.

Plugging $x = -1$ in the above equation, $2 = -2C$. So, $C = -1$.

Comparing the coefficients of x^2, $3 = A + B$. So, $B = 1$.

So,

$$\begin{aligned}\int \frac{2x^3+5x^2+x}{x^3+x^2-x-1}\,dx &= \int 2 + \frac{2}{x-1} + \frac{1}{(x+1)} - \frac{1}{(x+1)^2}\,dx \\ &= 2x + 2\ln|x-1| + \ln|x+1| + \frac{1}{x+1} + C.\end{aligned}$$

(b)

Note that $\sin^5 x = \sin^4 x \cdot \sin x = (1-\cos^2 x)^2 \sin x$. So, letting $u = \cos x$, we have $du = -\sin x dx$ and

$$\begin{aligned}\int \sin^5 x \cos^2 x\,dx &= \int (1-\cos^2 x)^2 \sin x \cos^2 x\,dx \\ &= -\int (1-u^2)^2 u^2 du \\ &= -\int u^6 - 2u^4 + u^2 du \\ &= -\left(\frac{u^7}{7} - 2\cdot\frac{u^5}{5} + \frac{u^3}{3}\right) + C \\ &= -\frac{1}{7}\cos^7 x + \frac{2}{5}\cos^5 x - \frac{1}{3}\cos^3 x + C.\end{aligned}$$

5. (20 points)

A spring is attached to a wall. In its resting position, the end of the spring is 1 m away from the wall. It takes 16 J of work to pull the spring so that the end is 3 m away from the wall. If the spring is brought back to rest, how much work does it then take to pull its end to 6 m away from the wall?

Solution: Let x be the distance that the spring is stretched from the resting position, and let k be the spring constant. By Hooke's law we have $F = kx$. So

$$\text{Work} \quad = \quad \int_0^{3-1} kx\,dx = \frac{kx^2}{2}\bigg|_0^2 = \frac{k(2)^2}{2} = 2k\ J$$

and the work is 16 J, so $k = 8$. To pull the end to 6 m away from the wall, this is 5 m from rest, so it takes

$$\text{Work} \quad = \quad \int_0^5 8x\,dx = 4x^2\bigg|_0^5 = 4(5)^2 = 4(25) = 100\ J$$