# Math 162: Calculus IIA

Final Exam

May 2nd, 2016

**Please circle your section:**

**Gage MW 2:00pm** **Harper TR 9:40am** **Lubkin MWF 9:00am**

**Lungstrum MW 3:25pm** **Neuman TR 4:50pm** **Tucker MWF 10:25am**

**NAME (please print legibly):** SOLUTIONS

**Your University ID Number:** ________

**Your University E-mail:** ________

## Pledge of Honesty

I affirm that I will not give or receive any unauthorized help on this exam and that all work will be my own.

**Signature:** ________

| Part A | | |
|---|---|---|
| QUESTION | VALUE | SCORE |
| 1 | 10 | |
| 2 | 10 | |
| 3 | 15 | |
| 4 | 15 | |
| 5 | 15 | |
| 6 | 15 | |
| 7 | 20 | |
| TOTAL | 100 | |

| Part B | | |
|---|---|---|
| QUESTION | VALUE | SCORE |
| 8 | 15 | |
| 9 | 15 | |
| 10 | 10 | |
| 11 | 15 | |
| 12 | 10 | |
| 13 | 15 | |
| 14 | 10 | |
| 15 | 10 | |
| TOTAL | 100 | |

Instructions:

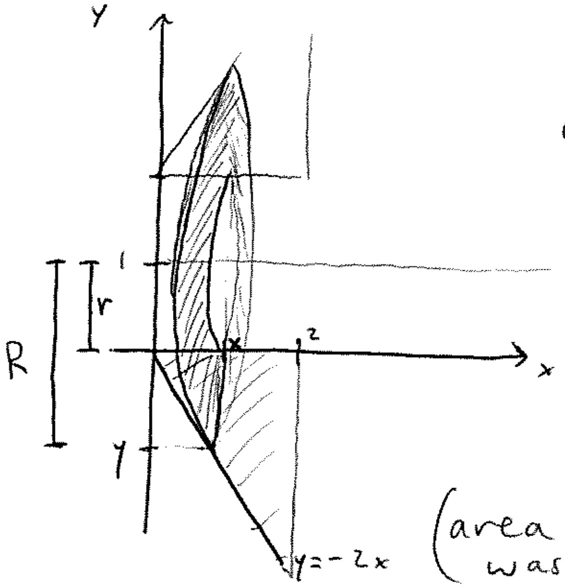
- **The use of calculators, cell phones, iPods, and other electronic devices at this exam is strictly forbidden. You must be physically separated from your cell phone.**
- **Show your work and justify your answers. You may not receive full credit for a correct answer if insufficient work is shown or insufficient justification is given.**
- **Put your answers in the spaces provided.**
- **You are responsible for checking that this exam has all 17 pages.**

Formulas:

- $\sin(x)\cos(x) = \frac{1}{2}\sin(2x)$
- $\sin^2(\theta) + \cos^2(\theta) = 1$
- $\tan^2(\theta) + 1 = \sec^2(\theta)$
- $\cos^2(\theta) = \frac{1}{2}(1 + \cos(2\theta))$
- $\sin^2(\theta) = \frac{1}{2}(1 - \cos(2\theta))$
- $\int \tan(x)dx = \ln|\sec(x)| + C$
- $\int \sec(x)dx = \ln|\sec(x) + \tan(x)| + C$
- $\int \frac{dx}{x^2+a^2} = \frac{1}{a}\arctan\left(\frac{x}{a}\right) + C$
- $\int \sec^3(x)dx = \sec(x)\tan(x) + \ln|\sec(x) + \tan(x)| + C$

**Part A**

**1. (10 points)** The region between the $x$-axis and the curve $y = -2x$ for $0 \le x \le 2$ is rotated about the line $y = 1$. Compute the volume.




(washer method)

area of washer $= \pi[R^2 - r^2]$

$r = 1$

$R = 1 - y$ (for $y \le 0$)

$= 1 - (-2x)$ $\left(\begin{matrix}\text{since}\\ y=-2x\end{matrix}\right)$, $0 \le x \le 2$.

$= 1 + 2x$

$\left(\begin{matrix}\text{area of}\\ \text{washer}\end{matrix}\right) = \pi\left[(1+2x)^2 - 1^2\right] = \pi\left[1 + 4x + 4x^2 - 1\right]$

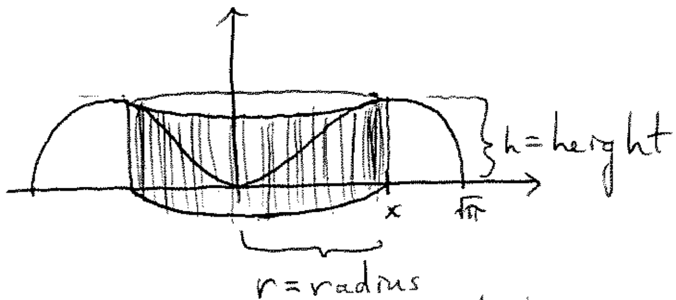
$= 4\pi(x + x^2)$.

$$V = \int_0^2 4\pi(x + x^2)\,dx = 4\pi\left(\frac{x^2}{2} + \frac{x^3}{3}\Big|_0^2\right)$$

$$= 4\pi\left(\frac{2^2}{2} + \frac{2^3}{3}\right) = 4\pi\left(\frac{6+8}{3}\right) = 4\pi\left(\frac{14}{3}\right) = \boxed{\frac{56\pi}{3}}$$

**2. (10 points)** The region between the $x$-axis and the curve $y = \sin(x^2)$ for $0 \leq x \leq \sqrt{\pi}$ is rotated about the $y$-axis. Compute the volume.

(looks $\approx$ like $\sin x$)




(cylindrical shells)

$$\begin{cases} r = x \\ h = y = \sin(x^2) \quad \text{for } 0 \leq x \leq \sqrt{\pi}. \end{cases}$$

(u-sub)

$u = x^2$

$du = 2x\,dx$

$x = 0 \Rightarrow u = 0^2 = 0$

$x = \sqrt{\pi} \Rightarrow u = (\sqrt{\pi})^2 = \pi$

$$V = \int_0^{\sqrt{\pi}} (2\pi r h)\,dx = 2\pi \int_0^{\sqrt{\pi}} x \cdot \sin(x^2)\,dx$$

$$= \pi \int_0^{\pi} \sin u \, du$$

$$= \pi\left(-\cos u \Big|_0^{\pi}\right) = \pi\left(-(-1) + 1\right) = \boxed{2\pi}$$

**3. (15 points)**

(a) Compute the indefinite integral $\int \tan^3(x)\sec^4(x)\,dx$.

$$\tan^3 x \cdot \sec^4 x = \tan^3 x \cdot \sec^2 x \cdot \sec^2 x$$
$$= \tan^3 x\,(1+\tan^2 x)\cdot \sec^2 x$$
$$= (\tan^3 x + \tan^5 x)\cdot \sec^2 x$$

So $\int \tan^3 x \cdot \sec^4 x\,dx = \int (\tan^3 x + \tan^5 x)\cdot \sec^2 x\,dx$

(u-sub)
$u = \tan x$
$du = \sec^2 x\,dx$

$$= \int (u^3 + u^5)\,du$$
$$= \frac{1}{4}u^4 + \frac{1}{6}u^6 + C$$
$$= \boxed{\frac{1}{4}\tan^4 x + \frac{1}{6}\tan^6 x + C}$$

(b) Find the average value of the function $f(x) = \tan^3(x)\sec^4(x)$ on the interval $[0, \frac{\pi}{3}]$.

$$f_{ave} = \frac{1}{\frac{\pi}{3} - 0}\int_0^{\frac{\pi}{3}} \tan^3 x \cdot \sec^4 x\,dx = \frac{3}{\pi}\left[\frac{1}{4}\tan^4 x + \frac{1}{6}\tan^6 x \Big|_0^{\frac{\pi}{3}}\right]$$
$$= \frac{3}{\pi}\left[\frac{1}{4}\tan^4\left(\frac{\pi}{3}\right) + \frac{1}{6}\tan^6\left(\frac{\pi}{3}\right)\right] = \frac{3}{\pi}\left[\frac{1}{4}(\sqrt{3})^4 + \frac{1}{6}(\sqrt{3})^6\right]$$
$$= \frac{3}{\pi}\left[\frac{1}{4}(3)^2 + \frac{1}{6}(3)^3\right] = \frac{3}{\pi}\left[\frac{27}{4}\right] = \boxed{\frac{81}{4\pi}}$$

**4. (15 points)**

(a) Find the following indefinite integral.

$$\int \frac{dx}{x^2 - 8x + 25}$$

(note: partial fractions won't help, since $x^2 - 8x + 25$ is irreducible)

Complete the square: $x^2 - 8x + 25 = (x - \frac{8}{2}) - (\frac{8}{2})^2 + 25 = (x-4)^2 + 3^2$

So $\displaystyle \int \frac{dx}{x^2-8x+25} = \int \frac{dx}{(x-4)^2+3^2} = \frac{1}{3^2}\int \frac{dx}{(\frac{x-4}{3})^2+1}$

(sub)

$\frac{x-4}{3} = \tan\theta$

$\frac{1}{3}dx = \sec^2\theta \, d\theta$

$\theta = \arctan(\frac{x-4}{3})$

$$= \frac{1}{3^2}\int \frac{3\sec^2\theta \, d\theta}{\tan^2\theta + 1} = \frac{3}{3^2}\int \frac{\cancel{\sec^2\theta}}{\cancel{\sec^2\theta}} \, d\theta$$

$$= \frac{1}{3}\int d\theta = \frac{1}{3}\theta + C$$

$$= \boxed{\frac{1}{3}\arctan\left(\frac{x-4}{3}\right) + C}$$

(b) Set up the partial fraction decomposition for the following integral in terms of variables, but do not solve for those variables.

$$\int \frac{x^2 - 8x + 25}{x(x^3 + x)}\, dx$$

degree 2

degree 4

2<4, so we don't need to do long division

$$\int \frac{x^2-8x+25}{x(x^3+x)}\,dx = \int \frac{x^2-8x+25}{x^2\underbrace{(x^2+1)}_{\text{irreducible}}}\,dx = \boxed{\int \left(\frac{A}{x} + \frac{B}{x^2} + \frac{Cx+D}{x^2+1}\right) dx}$$

**5. (15 points)** Consider the parametric curve defined by

$$\frac{dx}{dt} = 2t$$

$$x(t) = t^2 + 2$$
$$y(t) = t^3 - t$$

$$\frac{dy}{dt} = 3t^2 - 1$$

(a) Find two different tangent lines to the curve at the point $(3, 0)$.

$\frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)} = \frac{3t^2-1}{2t}$ ; we're @ $(x,y) = (3,0)$ when $\begin{cases} 3 = x = t^2+2 \\ 0 = y = t^3 - t \end{cases} \Longrightarrow t^2 = 1$, so $t = \pm 1$.

- $(t=1)$ line with slope $\left.\frac{dy}{dx}\right|_{t=1} = \frac{3(1^2)-1}{2(1)} = 1$, through the point $(3,0)$
  $\hookrightarrow y - 0 = (1)(x-3)$, so $\boxed{y = x - 3}$

- $(t=-1)$ line with slope $\left.\frac{dy}{dx}\right|_{t=-1} = \frac{3(-1)^2-1}{2(-1)} = -1$, through $(3,0)$
  $\hookrightarrow y - 0 = (-1)(x-3)$
  or $\boxed{y = -x + 3}$

(b) For which values of $t$ does the curve have a vertical tangent line?

Happens when $2t = 0$ and $3t^2 - 1 \neq 0$,
since $\frac{dy}{dx} = \frac{3t^2-1}{2t}$.

So $2t = 0 \Longrightarrow \boxed{t = 0}$. It could also happen where $2t = 0$ and $3t^2 - 1 = 0$, but no $t$ satisfies both equations.

(Hence just $t = 0$.)

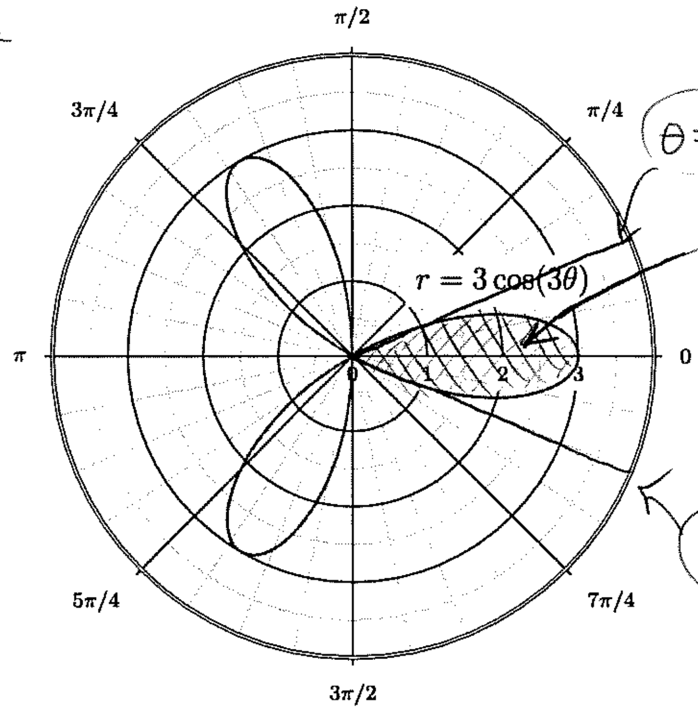
(c) Set up, but do **NOT** evaluate, an integral to calculate the arc length of the curve from $t = 0$ to $t = 4$.

$$L = \int_0^4 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \boxed{\int_0^4 \sqrt{(2t)^2 + (3t^2-1)^2}\, dt}$$

**6. (15 points)**

Use the polar area formula to find the area of one leaf of the three-leafed rose $r = 3\cos 3\theta$.

- Each leaf has the same area.
- Find the area of one leaf and multiply by 3.




$\theta = \frac{\pi}{6}$

Find the interval of $\theta$ for this leaf by setting $r=0$ and solving for $\theta$.

$\theta = -\frac{\pi}{6}$

$$0 = r = 3\cos(3\theta) \longrightarrow \cos(3\theta) = 0$$

$$\longrightarrow 3\theta = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}, \text{etc} \ldots$$

$$\theta = \pm\frac{\pi}{6}, \pm\frac{\pi}{2}, \pm\frac{5\pi}{6}, \text{etc.}$$

(these correspond to the shaded leaf)

$$3\left(\text{area of one leaf}\right) = 3\int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} \frac{1}{2} r^2 \, d\theta = \frac{3}{2}\int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} (3\cos(3\theta))^2 \, d\theta$$

$$= \frac{27}{2}\int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} \cos^2(3\theta)\, d\theta$$

(since $\cos^2(3\theta)$ is even) (half-angle identity)

$$= 27\int_0^{\frac{\pi}{6}} \cos^2(3\theta)\, d\theta = \frac{27}{2}\int_0^{\frac{\pi}{6}} (1 + \cos(6\theta))\, d\theta$$

$$= \frac{27}{2}\left[\theta + \frac{1}{6}\sin(6\theta)\Big|_0^{\frac{\pi}{6}}\right] = \frac{27}{2}\left[\frac{\pi}{6} + \frac{1}{6}\cancel{\sin(\pi)}^{\,0}\right] = \frac{27\pi}{12}$$

$$= \boxed{\frac{9\pi}{4}}$$

**7. (20 points)** Determine whether each of the following sequences converge. If a sequence converges find its limit; if it diverges, explain why.

(a) $a_n = \dfrac{2n}{3n+1}$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{2n}{3n+1}\cdot\frac{\left(\frac{1}{n}\right)}{\left(\frac{1}{n}\right)} = \lim_{n\to\infty}\frac{2}{3+\frac{1}{n}} = \boxed{\frac{2}{3}} \text{ (converges)}$$

(b) $a_n = \ln(2n) - \ln(3n+1)$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \ln\left(\frac{2n}{3n+1}\right).$$

Since $\dfrac{2n}{3n+1} \longrightarrow \dfrac{2}{3}$ as $n\to\infty$, and since $\ln(x)$ is continuous at $x = \frac{2}{3}$, we get $\lim\limits_{n\to\infty} a_n = \boxed{\ln\left(\frac{2}{3}\right)}$ (converges)

(c) $a_n = \dfrac{n}{\ln(n+1)}$

Indeterminate: $\frac{\infty}{\infty}$

$$\lim_{n\to\infty}\frac{n}{\ln(n+1)} \overset{\text{(l'Hospital)}}{=} \lim_{n\to\infty}\frac{1}{\left(\frac{1}{n+1}\right)} = \lim_{n\to\infty}(n+1) = \infty. \text{ (diverges)}$$

(d) $a_n = \dfrac{\sin(n)}{\sqrt{n}}$ $\quad -1 \le \sin(n) \le 1 \Longrightarrow -\frac{1}{\sqrt{n}} \le \frac{\sin(n)}{\sqrt{n}} \le \frac{1}{\sqrt{n}}$

Since $\lim\limits_{n\to\infty}\left(\pm\frac{1}{\sqrt{n}}\right) = 0$, $\boxed{\lim\limits_{n\to\infty}\frac{\sin(n)}{\sqrt{n}} = 0}$ by the Squeeze Theorem. (converges)

(e) $a_n = \sin\left(\frac{1}{\sqrt{n}}\right)$

$\lim\limits_{n\to\infty}\frac{1}{\sqrt{n}} = 0$, and $\sin x$ is continuous at $x=0$.

So $$\lim_{n\to\infty}\sin\left(\frac{1}{\sqrt{n}}\right) = \sin\left(\lim_{n\to\infty}\frac{1}{\sqrt{n}}\right) = \sin(0) = \boxed{0} \text{ (converges)}$$

**Part B**

**8. (15 points)** Determine whether each of the following series diverges or converges. If a series diverges, justify. If it converges, justify, and find the value of its sum.

(a) $\sum_{n=0}^{\infty} \frac{2n}{3n+1}$

Since $\lim_{n\to\infty} \frac{2n}{3n+1} = \frac{2}{3} \neq 0$, the series diverges by the Divergence Test.

(b) $\sum_{n=0}^{\infty} \frac{2^n + (-2)^n}{3^n} = \sum_{n=0}^{\infty} \left( \frac{2^n}{3^n} + \frac{(-2)^n}{3^n} \right) = \sum_{n=0}^{\infty} \left[ \left(\frac{2}{3}\right)^n + \left(-\frac{2}{3}\right)^n \right]$

- $\sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^n = \frac{1}{1-\frac{2}{3}} = 3$ (convergent geometric series w/ $a=1, r=\frac{2}{3}$)
- $\sum_{n=0}^{\infty} \left(-\frac{2}{3}\right) = \frac{1}{1+\frac{2}{3}} = \frac{3}{5}$ (conv. geom. series w/ $a=1, r=-\frac{2}{3}$)

So $\sum_{n=0}^{\infty} \frac{2^n+(-2)^n}{3^n} = \sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^n + \sum_{n=0}^{\infty} \left(-\frac{2}{3}\right)^n = \boxed{3 + \frac{3}{5}}$ (convergent)

(c) $\sum_{n=1}^{\infty} \frac{2}{n(n+2)} = \sum_{n=1}^{\infty} \left( \frac{1}{n} - \frac{1}{n+2} \right)$ (partial fractions)

$$s_n = \sum_{i=1}^{n} \frac{2}{i(i+2)} = \sum_{i=1}^{n} \left( \frac{1}{i} - \frac{1}{i+2} \right)$$

$$= \left(1 - \cancel{\tfrac{1}{3}}\right) + \left(\tfrac{1}{2} - \cancel{\tfrac{1}{4}}\right) + \left(\cancel{\tfrac{1}{3}} - \cancel{\tfrac{1}{5}}\right) + \left(\cancel{\tfrac{1}{4}} - \cancel{\tfrac{1}{6}}\right) + \left(\cancel{\tfrac{1}{5}} - \cancel{\tfrac{1}{7}}\right) + \cdots + \left(\cancel{\tfrac{1}{n-2}} - \cancel{\tfrac{1}{n}}\right) + \left(\cancel{\tfrac{1}{n-1}} - \tfrac{1}{n+1}\right) + \left(\cancel{\tfrac{1}{n}} - \tfrac{1}{n+2}\right)$$

$$= 1 + \frac{1}{2} - \frac{1}{n+1} - \frac{1}{n+2} \longrightarrow 1 + \frac{1}{2} = \frac{3}{2} \text{ as } n \to \infty.$$

So the series converges and $\sum_{n=1}^{\infty} \frac{2}{n(n+2)} = \boxed{\frac{3}{2}}$

**9. (15 points)**

(a) Use the Integral Test to check whether the series $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)}$ converges or diverges.

$f(x) = \frac{1}{x(\ln x)}$ is positive & continuous. Is it decreasing?

- $f'(x) = \frac{-1}{[x(\ln x)]^2}(\ln x + 1) < 0$ if $x \geq 2$, say. $\Rightarrow$ decreasing.

So the Integral Test applies.

$$\int_2^\infty \frac{1}{x(\ln x)}dx = \lim_{r\to\infty} \int_2^r \frac{1}{x(\ln x)}dx = \lim_{r\to\infty} \int_{\ln 2}^{\ln r} \frac{1}{u}du = \lim_{r\to\infty}\left(\ln u \Big|_{\ln 2}^{\ln r}\right)$$

$$= \lim_{r\to\infty}\left(\ln(\ln r) - \ln(\ln 2)\right) = \infty.$$

(u-sub)
$u = \ln x$
$du = \frac{1}{x}dx$

Since the integral diverges, the series diverges by the Integral Test.

(b) Determine all values of $p$ for which the series

$$\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^p}$$

converges. ($p=1 \Rightarrow$ diverges, by part (a))

($p \neq 1$) $g(x) = \frac{1}{[x(\ln x)^p]}$ continuous & positive for $x \geq 2$.

- $g'(x) = -\frac{1}{[x(\ln x)^p]^2}\left((\ln x)^p + p\cdot(\ln x)^{p-1}\right) < 0 \Leftrightarrow (\ln x)^p + p\cdot(\ln x)^{p-1} > 0$

(divide by $(\ln x)^{p-1}$) $\Leftrightarrow \ln x + p > 0$

$\Leftrightarrow x > e^{-p}$.

- So $g$ is eventually decreasing, which is all we need for the integral test.

$$\int_2^\infty \frac{1}{x(\ln x)^p}dx = \lim_{r\to\infty}\int_2^r \frac{1}{x(\ln x)^p}dx = \lim_{r\to\infty}\int_{\ln 2}^{\ln r}\frac{1}{u^p}du = \lim_{r\to\infty}\left(\frac{u^{1-p}}{1-p}\Big|_{\ln 2}^{\ln r}\right)$$

$$= \lim_{r\to\infty}\left(\frac{(\ln r)^{1-p}}{1-p} - \frac{(\ln 2)^{1-p}}{1-p}\right) = \begin{cases} \text{diverges} & \text{if } p<1, \\ -\frac{(\ln 2)^{1-p}}{1-p} & \text{if } p>1. \end{cases}$$

(u-sub)
$u = \ln x$
$du = \frac{1}{x}dx$

So the series converges precisely when $\boxed{p>1}$, by the Integral Test.

**10. (10 points)** Determine if the following series converge or diverge and justify your answer.

(a) $\sum_{n=1}^{\infty} \frac{\ln(n)}{n}$ Note: $\ln(n) \geq 1 \iff n \geq e \approx 2.7\ldots$ (so $n \geq 3$, say)

So $\frac{\ln(n)}{n} \geq \frac{1}{n} > 0$ for $n \geq 3$. Since $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges (harmonic series), the Comparison Test says that $\sum_{n=1}^{\infty} \frac{\ln(n)}{n}$ also diverges.

(b) $\sum_{n=1}^{\infty} \left(\frac{2n}{3n+1}\right)^n$ (root test!)

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \sqrt[n]{\left(\frac{2n}{3n+1}\right)^n} = \lim_{n\to\infty} \left(\frac{2n}{3n+1}\right)^{n\cdot\frac{1}{n}} = \lim_{n\to\infty} \frac{2n}{3n+1}$$

$= \frac{2}{3} < 1$. Hence the series converges by the root test.

(c) $\sum_{n=1}^{\infty} \frac{\arctan(n)}{n^{1.2}}$ (limit comparison) $\begin{cases} a_n = \frac{\arctan(n)}{n^{1.2}}, \\ b_n = \frac{1}{n^{1.2}}. \end{cases}$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\arctan(n)}{\cancel{n^{1.2}}} \cdot \cancel{n^{1.2}} = \lim_{n\to\infty} \arctan(n) = \frac{\pi}{2} > 0.$$

Since the limit exists and is nonzero, the limit comparison test says $\sum a_n$ and $\sum b_n$ both converge or both diverge. But $\sum b_n = \sum_{n=1}^{\infty} \frac{1}{n^{1.2}}$ is a convergent p-series ($p = 1.2 > 1$), so $\sum a_n = \sum_{n=1}^{\infty} \frac{\arctan(n)}{n^{1.2}}$ converges.

**11. (15 points)** Decide whether the following series are absolutely convergent, conditionally convergent, or divergent, and justify.

(a) $\sum_{n=1}^{\infty}(-1)^n \frac{n}{\sqrt{n^3+2}}$

(AST) $\lim_{n\to\infty} \frac{n}{\sqrt{n^3+2}} = \lim_{n\to\infty} \frac{n}{\sqrt{n^3+2}} \cdot \frac{(\frac{1}{n})}{(\frac{1}{n})} = \lim_{n\to\infty} \frac{1}{\sqrt{n+2/n^2}} = 0$ ✓

Decreasing? $f(x) = \frac{x}{\sqrt{x^3+2}} \Longrightarrow f'(x) = \frac{\sqrt{x^3+2} - \frac{x}{2\sqrt{x^3+2}}(3x^2)}{x^3+2}$

$= \frac{(x^3+2-\frac{3}{2}x^3)}{(x^3+2)^{3/2}} = \frac{2-\frac{1}{2}x^3}{(x^3+2)} < 0$ ✓

if $x^3 > 4 \longrightarrow x > 4^{1/3}$ (so $x \geq 2$, say)

So the series converges by AST.

Is it absolutely convergent?

$\sum_{n=1}^{\infty}\left|(-1)^n \frac{n}{\sqrt{n^3+2}}\right| = \sum_{n=1}^{\infty} \frac{n}{\sqrt{n^3+2}}$ ; limit comparison w/ $\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}$ (divergent p-series $p=\frac{1}{2}<1$)

$\lim_{n\to\infty} \frac{\left(\frac{n}{\sqrt{n^3+2}}\right)}{\left(\frac{1}{\sqrt{n}}\right)} = \lim_{n\to\infty} \frac{n^{3/2}}{\sqrt{n^3+2}} \cdot \frac{(1/n^{3/2})}{(1/n^{3/2})} = \lim_{n\to\infty} \frac{1}{\sqrt{1+2/n^3}} = 1 > 0$.

The limit exists and is nonzero. Since $\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}$ diverges, ~~our series~~ $\sum_{n=1}^{\infty}\frac{n}{\sqrt{n^3+2}}$ also diverges (by limit comparison). So the series $\sum_{n=1}^{\infty}(-1)^n\frac{n}{\sqrt{n^3+2}}$ is conditionally convergent.

(b) $\sum_{n=1}^{\infty}(-1)^n \frac{n!}{(2n)!}$

(ratio test)

$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{(n+1)!}{(2(n+1))!} \cdot \frac{(2n)!}{n!} = \lim_{n\to\infty} \frac{(n+1)!}{n!} \cdot \frac{(2n)!}{(2n+2)!}$

$= \lim_{n\to\infty} \frac{(n+1)\cdot \cancel{(n!)}}{\cancel{n!}} \cdot \frac{\cancel{(2n)!}}{(2n+2)(2n+1)\cdot \cancel{(2n)!}}$

$= \lim_{n\to\infty} \frac{n+1}{(2n+2)(2n+1)} = 0 < 1$

So $\sum_{n=1}^{\infty}(-1)^n \frac{n!}{(2n)!}$ converges absolutely by the Ratio Test.

**12. (10 points)** Find the radius of convergence and the interval of convergence of the following power series.

(a) $\sum_{n=1}^{\infty} \frac{4^n}{n!}(x+1)^n$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{4^{n+1}}{(n+1)!}(x+1)^{n+1}\cdot\frac{n!}{4^n\cdot(x+1)^n}\right|$$

$$= \lim_{n\to\infty}\frac{4^{n+1}}{4^n}\cdot\frac{n!}{(n+1)!}\left|\frac{(x+1)^{n+1}}{(x+1)^n}\right|$$

$$= \lim_{n\to\infty}\frac{4}{n+1}|x+1| = 0 < 1$$

for $\underline{\text{any}}$ $x$.

So the radius of convergence is $\boxed{\infty}$ and the interval of convergence is $\boxed{(-\infty,\infty)}$.

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(b) $\sum_{n=1}^{\infty} \frac{(2x-1)^n}{5^n\sqrt{n}}$

$$\lim_{n\to\infty}\left|\frac{(2x-1)^{n+1}}{5^{n+1}\cdot\sqrt{n+1}}\cdot\frac{5^n\sqrt{n}}{(2x-1)^n}\right| = \lim_{n\to\infty}\left|\frac{(2x-1)^{n+1}}{(2x-1)^n}\right|\frac{5^n}{5^{n+1}}\cdot\frac{\sqrt{n}}{\sqrt{n+1}}$$

$$= \lim_{n\to\infty}|(2x-1)|\frac{1}{5}\sqrt{\frac{n}{n+1}}$$

$$= \frac{1}{5}|2x-1|$$

Diverges if $\frac{1}{5}|2x-1|>1$

Converges if $\frac{1}{5}|2x-1|<1 \longrightarrow |2x-1|<5$

$\hookrightarrow -5 < 2x-1 < 5$

$\hookrightarrow -4 < 2x < 6$

$\hookrightarrow -2 < x < 3$

- Radius of convergence is $\frac{3-(-2)}{2} = \boxed{\frac{5}{2}}$

Check endpoints:

@ $\underline{x=3}$: $\sum_{n=1}^{\infty}\frac{5^n}{5^n\sqrt{n}} = \sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}$ (divergent p-series, $p=\frac{1}{2}$)

@ $\underline{x=-2}$: $\sum_{n=1}^{\infty}\frac{(-5)^n}{5^n\sqrt{n}} = \sum_{n=1}^{\infty}\frac{(-1)^n}{\sqrt{n}}$ (convergent by AST)

So the interval of convergence is $\boxed{[-2,3)}$.

**13. (15 points)**

(a) Find the Taylor series centered at $a = 0$ of the function $\ln(1 - x^2)$ and its interval of convergence.

Note: $\frac{d}{dx}\ln(1-x^2) = \frac{1}{1-x^2}(-2x) = -2x\sum_{n=0}^{\infty}(x^2)^n$

$= -2\sum_{n=0}^{\infty} x^{2n+1}$ if $|x^2|<1$

$\hookrightarrow |x|<1$.

So $\ln(1-x^2) = \int\left(\frac{d}{dx}\ln(1-x^2)\right)dx = \int\left(-2\sum_{n=0}^{\infty}x^{2n+1}\right)dx$

$= -2\sum_{n=0}^{\infty}\frac{x^{2n+2}}{2n+2} + C = -\sum_{n=0}^{\infty}\frac{x^{2n+2}}{n+1} + C$

Plug in $x=0$: $\ln(1) = -\sum_{n=0}^{\infty}\frac{0^{2n+2}}{n+1} + C = C \Rightarrow C=0$.

So the Taylor series @ $a=0$ is $\boxed{-\sum_{n=0}^{\infty}\frac{x^{2n+2}}{n+1}}$ with interval of convergence $\boxed{(-1,1)}$

(note: it diverges when $x=\pm 1$)

(b) Write the integral $\int_0^x \ln(1 - t^2)dt$ as a power series in $x$, and find its interval of convergence.

$\int_0^x \ln(1-t^2)\,dt = \int_0^x\left(-\sum_{n=0}^{\infty}\frac{t^{2n+2}}{n+1}\right)dt$

$= -\sum_{n=0}^{\infty}\frac{t^{2n+3}}{(n+1)(2n+3)}\Big|_0^x$

$= \boxed{-\sum_{n=0}^{\infty}\frac{x^{2n+3}}{(n+1)(2n+3)}}$ if $|x|<1 \Leftrightarrow -1<x<1$.

@ $x=1$: $-\sum_{n=0}^{\infty}\frac{1}{(n+1)(2n+3)}$; limit comparison with $\sum_{n=1}^{\infty}\frac{1}{n^2}$ (convergent p-series, $p=2$)

$\lim_{n\to\infty}\frac{1}{(n+1)(2n+3)}\cdot\frac{1}{\left(\frac{1}{n^2}\right)} = \lim_{n\to\infty}\frac{n^2}{(n+1)(2n+3)} = \frac{1}{2}>0 \Rightarrow -\sum_{n=0}^{\infty}\frac{1}{(n+1)(2n+3)}$ converges.

@ $x=-1$: $-\sum_{n=0}^{\infty}\frac{(-1)^{2n+3}}{(n+1)(2n+3)} = \sum_{n=0}^{\infty}\frac{1}{(n+1)(2n+3)}$ converges as above.

So the interval of convergence is $\boxed{[-1,1]}$.

**14. (10 points)**

(a) Compute the Taylor series of the function $f(x) = \sin(x)$ centered at $a = -\pi/2$ and its interval of convergence.

| | |
|---|---|
| $f^{(0)}(x) = \sin x$ | $f^{(0)}(-\frac{\pi}{2}) = -1$ |
| $f^{(1)}(x) = \cos x$ | $f^{(1)}(-\frac{\pi}{2}) = 0$ |
| $f^{(2)}(x) = -\sin x$ | $f^{(2)}(-\frac{\pi}{2}) = 1$ |
| $f^{(3)}(x) = -\cos x$ | $f^{(3)}(-\frac{\pi}{2}) = 0$ |
| $f^{(4)}(x) =$ (repeats) | (repeats) |

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(-\frac{\pi}{2})}{n!}\left(x - \left(-\frac{\pi}{2}\right)\right)^n$$

So $\begin{cases} f^{(2n)}(-\frac{\pi}{2}) = (-1)^{n+1} & \text{(even terms)} \\ f^{(2n+1)}(-\frac{\pi}{2}) = 0 & \text{(odd terms)} \end{cases}$

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(-\frac{\pi}{2})}{n!}\left(x - \left(-\frac{\pi}{2}\right)\right)^n = \boxed{\sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{(2n)!}\left(x + \frac{\pi}{2}\right)^{2n}}$$

(ratio test)

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(-1)^{n+2}}{(2n+2)!}\left(x+\frac{\pi}{2}\right)^{2n+2} \cdot \frac{(2n)!}{(-1)^{n+1}\cdot\left(x+\frac{\pi}{2}\right)^{2n}}\right|$$

$$= \lim_{n\to\infty} \frac{(2n)!}{(2n+2)!}\left|\left(x+\frac{\pi}{2}\right)^2\right| = \lim_{n\to\infty} \frac{\left|\left(x+\frac{\pi}{2}\right)^2\right|}{(2n+2)(2n+1)} = 0 < 1$$

for any $x$.

So the interval of convergence is $\boxed{(-\infty, \infty)}$.

(b) Find the Taylor polynomials of $f(x)$ of degree 2 and 3.

degree 2: $T_2(x) = \sum_{i=0}^{2} \frac{f^{(i)}(-\frac{\pi}{2})}{i!}\left(x+\frac{\pi}{2}\right)^i = \underbrace{-1}_{i=0} + \underbrace{0}_{i=1} + \underbrace{\frac{1}{2!}\left(x+\frac{\pi}{2}\right)^2}_{i=2}$

$= \boxed{-1 + \frac{1}{2}\left(x+\frac{\pi}{2}\right)^2}$

degree 3: Since $f^{(3)}(-\frac{\pi}{2}) = 0$,

$T_3(x) = T_2(x) + 0 = \boxed{-1 + \frac{1}{2}\left(x+\frac{\pi}{2}\right)^2}$

**15. (10 points)**

(a) Find the first five nonzero terms of the Maclaurin series expansion of the function

$$f(x) = \frac{\cos(x^2) - (1 - \frac{x^4}{2})}{x^8}.$$

$$\cos(x^2) = \sum_{n=0}^{\infty} \frac{(-1)^n \cdot (x^2)^{2n}}{(2n)!} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{4n}$$

$$= 1 - \frac{x^4}{2!} + \frac{x^8}{4!} - \frac{x^{12}}{6!} + \frac{x^{16}}{8!} - \frac{x^{20}}{10!} + \frac{x^{24}}{12!} - \cdots$$

$$\text{So } f(x) = \frac{1}{x^8}\left[\left(\cancel{1} - \cancel{\frac{x^4}{2!}} + \frac{x^8}{4!} - \frac{x^{12}}{6!} + \frac{x^{16}}{8!} - \frac{x^{20}}{10!} + \frac{x^{24}}{12!} - \cdots\right) - \left(\cancel{1} - \cancel{\frac{x^4}{2}}\right)\right]$$

$$\boxed{= \frac{1}{4!} - \frac{x^4}{6!} + \frac{x^8}{8!} - \frac{x^{12}}{10!} + \frac{x^{16}}{12!} - \cdots}$$

(b) What is the value of $f^{(10)}(0)$?

The coefficient of $x^{10}$ is $0$ (i.e. not there),

$$\text{so } \frac{f^{(10)}(0)}{10!} = 0 \Longrightarrow \boxed{f^{(10)}(0) = 0.}$$

(c) What is the value of $f^{(12)}(0)$?

(coefficient of $x^{12}$)

$$\frac{f^{(12)}(0)}{12!} = -\frac{1}{10!} \Longrightarrow f^{(12)}(0) = -\frac{12!}{10!} = -(12)(11)$$

$$\boxed{= -132}$$

(d) What is the value of $\lim_{x \to 0} f(x)$?

$$\lim_{x \to 0} f(x) = \lim_{x \to 0}\left(\frac{1}{4!} - \frac{x^4}{6!} + \frac{x^8}{8!} - \cdots\right) = \frac{1}{4!} = \boxed{\frac{1}{24}}$$