1. (15 points) Determine if the following integrals are convergent or divergent.

(a) $\int_0^3 \frac{1}{(2-x)^2}\,dx = \int_0^2 \frac{1}{(2-x)^2}\,dx + \int_2^3 \frac{1}{(2-x)^2}\,dx$.

$$\int_0^2 \frac{1}{(2-x)^2}\,dx = \lim_{t\to 2^-} \int_0^t \frac{1}{(2-x)^2}\,dx = \lim_{t\to 2^-} \left[\frac{1}{2-x}\right]_0^t$$

$$= \lim_{t\to 2^-} \left(\frac{1}{2-t} - \frac{1}{2}\right) = \infty .$$

So, the integral is divergent .

(b) $\int_1^\infty e^{-x^2}\,dx$ (*Hint: does* $\int_1^\infty e^{-x}\,dx$ *converge?*)

Since $x \leq x^2$ for all $x \geq 1$,

$$e^{-x^2} \leq e^{-x}$$

for all $x \geq 1$.

From $\int_1^\infty e^{-x}\,dx = \lim_{t\to\infty} \int_1^t e^{-x}\,dx = \lim_{t\to\infty} \left[-e^{-x}\right]_1^t = \lim_{t\to\infty}\left(-e^{-t} + e^{-1}\right)$

$= e^{-1}$ (convergent) ,

$\int_1^\infty e^{-x^2}\,dx$ is convergent as well by the comparison test.

2. (15 points) Consider the curve $y = x^{3/2}$.

(a) Find the arc length function $s(t)$ starting at $x = 0$. (Hint: $s(t) =$ the arc length along $y = x^{3/2}$ from $x = 0$ to $x = t$.)

$$S(t) = \int_0^t \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int_0^t \sqrt{1 + \left(\frac{3}{2}x^{\frac{1}{2}}\right)^2}\, dx$$

$$= \int_0^t \sqrt{1 + \frac{9}{4}x}\, dx \qquad \begin{cases} u = 1 + \frac{9}{4}x \\ du = \frac{9}{4}dx \end{cases} \qquad \begin{cases} x=0 & u=1 \\ x=t & u = 1+\frac{9}{4}t \end{cases}$$

$$= \int_1^{1+\frac{9}{4}t} \sqrt{u}\ \frac{4}{9}du$$

$$= \frac{4}{9}\left[\frac{2}{3}u^{\frac{3}{2}}\right]_1^{1+\frac{9}{4}t} = \frac{8}{27}\left(\left(1+\frac{9}{4}t\right)^{\frac{3}{2}} - 1\right).$$

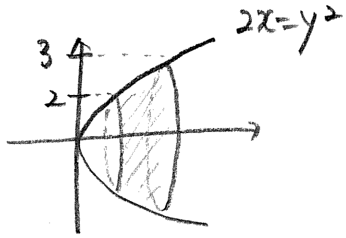
(b) Use part (a) to compute the length of the curve $y = x^{3/2}$ from $x = 0$ to $x = 2$. Simplify your answer completely.

$$S(2) = \frac{8}{27}\left(\left(1 + \frac{9}{4}\cdot 2\right)^{\frac{3}{2}} - 1\right)$$

$$= \frac{8}{27}\left(\left(\frac{11}{2}\right)^{\frac{3}{2}} - 1\right).$$

3. (15 points) Consider the curve C given by $2x = y^2$ for $2 \le y \le 3$.

(a) Find the surface area of the surface generated by rotating C about the x-axis. Evaluate the integral but do not simplify further.



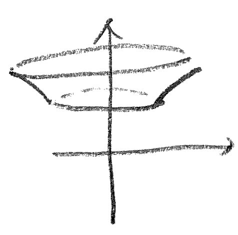
$$S = \int 2\pi y \, ds = \int_2^3 2\pi y \sqrt{1 + \left(\frac{dx}{dy}\right)^2} \, dy$$

$$= \int_2^3 2\pi y \sqrt{1 + y^2} \, dy \qquad \begin{cases} u = 1 + y^2 \\ du = 2y\,dy \end{cases}$$

$$= \int_5^{10} \pi \sqrt{u} \, du \qquad \begin{cases} y=2 & u=5 \\ y=3 & u=10 \end{cases}$$

$$= \pi \frac{2}{3}\left[u^{\frac{3}{2}}\right]_5^{10} = \frac{2}{3}\pi\left(10^{\frac{3}{2}} - 5^{\frac{3}{2}}\right).$$

(b) Express but **do not compute** the surface area of the surface generated by rotating C about the y-axis as an integral dx.



$$\begin{cases} y=2 \Rightarrow x=2 \\ y=3 \Rightarrow x=\frac{9}{2} \end{cases}$$

$$y = \sqrt{2x} \Rightarrow \frac{dy}{dx} = \sqrt{2} \cdot \frac{1}{2} x^{-\frac{1}{2}}$$

$$S = \int_2^{\frac{9}{2}} 2\pi x \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, dx = \int_2^{\frac{9}{2}} 2\pi x \sqrt{1 + \frac{1}{2x}} \, dx$$

(c) Express but **do not compute** the surface area of the surface generated by rotating C about the y-axis as an integral dy.

$$S = \int_2^3 2\pi x \sqrt{1 + \left(\frac{dx}{dy}\right)^2} \, dy = \int_2^3 2\pi \cdot \frac{y^2}{2} \sqrt{1 + y^2} \, dy$$

$$= \int_2^3 \pi y^2 \sqrt{1 + y^2} \, dy$$

4. (15 points) Consider the parametric curve $x = 1 + e^t$, $y = 2t - t^2$.

(a) Find the tangent line to the curve at the point $(2, 0)$.

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{2-2t}{e^t}$$

$$\begin{cases} x = 1+e^t = 2 \\ y = 2t - t^2 = 0 \end{cases} \Rightarrow t = 0$$

$$\left.\frac{dy}{dx}\right|_{(2,0)} = \frac{2-2\cdot 0}{e^0} = 2$$

$$(y-0) = 2(x-2)$$

$$\Leftrightarrow y = 2(x-2).$$

(b) Find a range of t values for which the curve is concave down.

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}} = \frac{\frac{d}{dt}\left(\frac{2-2t}{e^t}\right)}{e^t} = \frac{\frac{-2e^t-(2-2t)e^t}{e^{2t}}}{e^t} = \frac{-2-2+2t}{e^{2t}} = \frac{-4+2t}{e^{2t}} < 0$$

$$\Leftrightarrow -4+2t<0 \Leftrightarrow t<2$$

(c) Set up but **do not evaluate** an integral whose value is the area of the region bounded by this curve and the x-axis.

$$y=0 \Leftrightarrow 2t-t^2=0$$

$$\Leftrightarrow t(2-t)=0$$

$$\Leftrightarrow t=0,\ 2$$

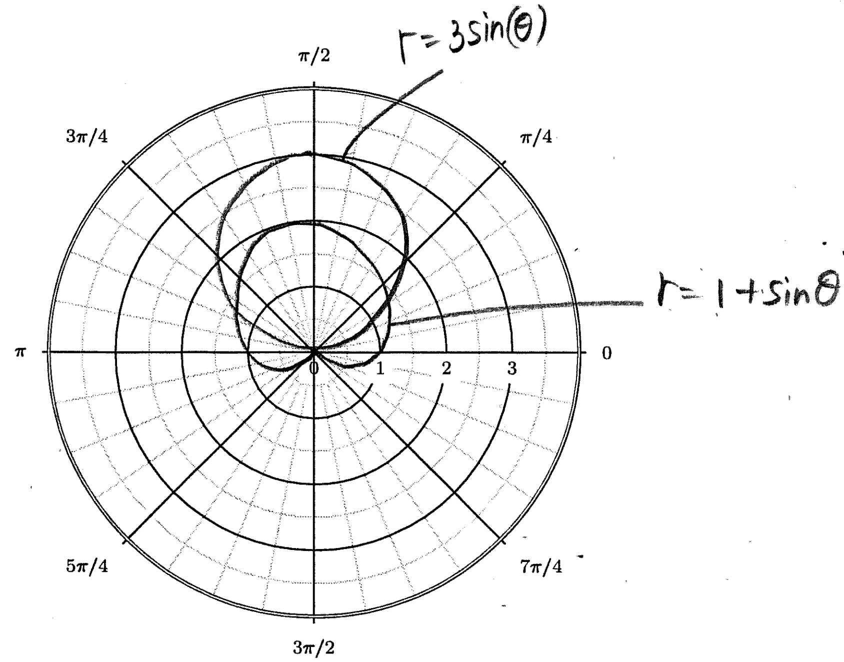
$$\text{Area} = \int_0^2 y\, dx$$

$$= \int_0^2 (2t-t^2)e^t\, dt$$

(d) Set up but **do not evaluate** an integral whose value is the arc length of this curve for t values $1 \le t \le 5$.

$$\int_1^5 \sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\, dt = \int_1^5 \sqrt{e^{2t}+(2-2t)^2}\, dt$$

5. (15 points) Consider the polar curves given by $r = 3\sin(\theta)$ and $r = 1 + \sin(\theta)$ for $0 \le \theta \le 2\pi$.



(a) Draw a clear sketch of both curves (with each curve clearly labeled) above.

(b) For which value(s) of θ do the curves intersect?

$3\sin\theta = 1 + \sin\theta$

$\Leftrightarrow 2\sin\theta = 1$

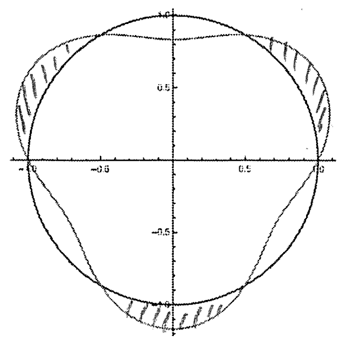
$\Leftrightarrow \sin\theta = \frac{1}{2}$

$\Leftrightarrow \theta = \frac{\pi}{6}, \frac{5}{6}\pi$

(c) Write down but **do not evaluate** an integral whose value gives the arc length of the perimeter of the region inside the curve $r = 3\sin(\theta)$, but outside the curve $r = 1+\sin(\theta)$.

$$\text{Arclength} = \int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \sqrt{(3\sin\theta)^2 + (3\cos\theta)^2}\, d\theta + \int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \sqrt{(1+\sin\theta)^2 + (\cos\theta)^2}\, d\theta$$

6. (15 points) On the picture below one graph is a circle and the other one has a polar equation $r(\theta) = 1 + \frac{1}{6}\sin(3\theta)$. Find the total area that lies inside $r(\theta) = 1 + \frac{1}{6}\sin(3\theta)$, but outside the circle. Simplify your answer completely.



① points of intersection

$1+\frac{1}{6}\sin(3\theta)=1 \iff \sin 3\theta = 0$

$\iff 3\theta = n\pi$

$\iff \theta = \frac{\pi}{3}n \;,\; n=0,\pm1,\cdots$

② $\text{Area} = 3\cdot\frac{1}{2}\int_0^{\frac{\pi}{3}} (1+\frac{1}{6}\sin(3\theta))^2 - 1^2 \, d\theta$

$= \frac{3}{2}\int_0^{\frac{\pi}{3}} \frac{1}{3}\sin(3\theta) + \frac{1}{36}\sin^2 3\theta \, d\theta$

$= \frac{3}{2}\cdot\frac{1}{3}\left[-\frac{\cos 3\theta}{3}\right]_0^{\frac{\pi}{3}} + \frac{3}{2}\cdot\frac{1}{36}\int_0^{\frac{\pi}{3}} \frac{1-\cos(6\theta)}{2} \, d\theta$

$= -\frac{1}{6}(\cos\pi - \cos 0) + \frac{1}{48}\left[\theta - \frac{\sin(6\theta)}{6}\right]_0^{\frac{\pi}{3}}$

$= \frac{1}{3} + \frac{1}{48}\left(\frac{\pi}{3}\right) = \frac{1}{3} + \frac{\pi}{144}$

7. (10 points) Determine whether each of the following sequences converges or diverges. If a sequence converges, justify and find its limit. If it diverges, state whether it diverges to ∞, to $-\infty$, or is oscillating.

(a) $a_n = \dfrac{2n+3}{5n-7}$ $= \dfrac{(2n+3)/n}{(5n-7)/n} = \dfrac{2+\frac{3}{n}}{5-\frac{7}{n}} \xrightarrow{n\to\infty} \dfrac{2+0}{5-0} = \dfrac{2}{5}$

converges to $\frac{2}{5}$.

(b) $a_n = \ln\left(\frac{1}{n}\right)$ $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} \ln\left(\frac{1}{n}\right) = \lim\limits_{x\to 0^+} \ln(x) = -\infty$

diverges to $-\infty$.

(c) $a_n = \cos(n)e^n$ Since $\cos(n)$ oscillates between -1 and 1 and e^n diverges to ∞, $\cos(n)e^n$ diverges and oscillates between $-\infty$ and ∞.

(d) $a_n = \dfrac{\cos(n)}{n}$ $0 \le |a_n| \le \frac{1}{n}$.

Since $0 \to 0$ and $\frac{1}{n} \to 0$ as $n\to\infty$, by squeeze thm, $a_n \xrightarrow{n\to\infty} 0$.

(e) $a_n = \dfrac{\ln(n)}{e^n}$

$\lim\limits_{x\to\infty} \dfrac{\ln x}{e^x} = \lim\limits_{x\to\infty} \dfrac{\frac{1}{x}}{e^x} = \lim\limits_{x\to\infty} \dfrac{1}{xe^x} = 0$ by l'Hopital's Thm. So, $a_n \xrightarrow{n\to\infty} 0$.