Math 162: Calculus IIA

Midterm II

March 29, 2016

Please circle your section:

Gage MW 2:00pm **Harper TR 9:40am** **Lubkin MWF 9:00am**

Lungstrum MW 3:25pm **Neuman TR 4:50pm** **Tucker MWF 10:25am**

NAME (please print legibly): SOLUTIONS

Your University ID Number: ____________________

Your University E-mail: ____________________

Pledge of Honesty

I affirm that I will not give or receive any unauthorized help on this exam and that all work will be my own.

Signature: ____________________

QUESTION	VALUE	SCORE
1	20	
2	10	
3	10	
4	20	
5	10	
6	10	
7	20	
TOTAL	100	

Instructions:

- **The use of calculators, cell phones, iPods and other electronic devices at this exam is strictly forbidden. You must be physically separated from your cell phone.**
- **Show your work and justify your answers. You may not receive full credit for a correct answer if insufficient work is shown or insufficient justification is given.**
- **Put your answers in the spaces provided.**
- **Justify and show all your work. No credit will be given if you do not justify and show all your work.**
- **You are responsible for checking that this exam has all 12 pages.**

Formulas:

- $\sin(x)\cos(x) = \frac{1}{2}\sin(2x)$
- $\sin^2(\theta) + \cos^2(\theta) = 1$
- $\tan^2(\theta) + 1 = \sec^2(\theta)$
- $\cos^2(\theta) = \frac{1}{2}(1 + \cos(2\theta))$
- $\sin^2(\theta) = \frac{1}{2}(1 - \cos(2\theta))$
- $\int \tan(x)dx = \ln|\sec(x)| + C$
- $\int \sec(x)dx = \ln|\sec(x) + \tan(x)| + C$
- $\int \frac{dx}{x^2+a^2} = \frac{1}{a}\arctan\left(\frac{x}{a}\right) + C$
- $\int \sec^3(x)dx = \frac{1}{2}[\sec(x)\tan(x) + \ln|\sec(x) + \tan(x)|] + C$

1. (20 points) For each of the following improper integrals, determine whether it converges or diverges. If it converges, find its value. If it diverges, explain why.

(a)

$$\int_1^\infty \frac{\pi}{x^2}\,dx$$

$$\lim_{T\to\infty} \int_1^T \frac{\pi}{x^2}\,dx = \lim_{T\to\infty} -\frac{\pi}{x}\Big]_{x=1}^{x=T}$$

$$= 0 - \left(-\frac{\pi}{1}\right)$$

$$= \pi$$

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(b)

$$\int_1^3 \frac{1}{x(x-2)}\, dx$$

$$= \int_1^3 \frac{1}{2}\left(\frac{1}{x-2} - \frac{1}{x}\right) dx$$

$$= \frac{1}{2}\left(\underbrace{\int_1^2 \frac{1}{x-2}\, dx}_{\lim_{b\to 2^-} \ln|x-2| \Big]_{x=1}^{x=b}} - \int_1^2 \frac{1}{x} + \int_2^3 \frac{1}{x-2}\, dx - \int_2^3 \frac{1}{x}\, dx\right)$$

$$\lim_{b\to 2^-} \ln|b-2| - 0$$

$$\longrightarrow -\infty$$

The integral diverges.

2. (10 points) Compare the integral $I = \int_1^\infty \frac{1}{x}\sqrt{1+\left(-\frac{1}{x^2}\right)^2}\,dx$ to $\int_1^\infty \frac{1}{x}\,dx$, and use the Comparison Test to determine whether I converges or diverges.

$$\frac{1}{x} \cdot \sqrt{1 + \frac{1}{x^4}} > \frac{1}{x} \cdot (1)$$

So

$$\int_1^\infty \frac{\sqrt{1 + 1/x^4}}{x}\,dx > \int_1^\infty \frac{1}{x}\,dx$$

$$= \ln|x| \Big]_{x=1}^{x\to\infty}$$

$$= \infty$$

Therefore, the integral diverges.

3. (10 points) Find the surface area of the surface of revolution generated by rotating the curve $y = \frac{x^3}{3}$ for $0 \leq x \leq 2$ around the x-axis.

$$\int_0^2 2\pi \cdot \left(\frac{x^3}{3}\right) \sqrt{1 + x^4}\, dx$$

$$\frac{2\pi}{3} \int_0^2 x^3 \sqrt{1 + x^4}\, dx$$

$$u = 1 + x^4$$

$$du = 4x^3\, dx$$

$$\frac{2\pi}{3 \cdot 4} \int_1^{17} u^{1/2}\, du$$

$$= \frac{\pi}{6}\left(\frac{2}{3}\left(17^{3/2} - 1\right)\right)$$

$$= \frac{\pi}{9}\left(17^{3/2} - 1\right)$$

4. (20 points) Consider the parametric curve defined by

$$x(t) = t^2$$
$$y(t) = 3t - t^3.$$

(a) Calculate $\dfrac{dy}{dx}$.

$$\frac{dy}{dx} = \frac{3 - 3t^2}{2t}$$

(b) For which values of t does the curve have a horizontal tangent line?

$$3 - 3t^2 = 0 \implies t = \pm 1$$

(c) For which values of t does the curve have a vertical tangent line?

$$2t = 0 \implies t = 0$$

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(d) Calculate $\frac{d^2y}{dx^2}$.

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}} = \frac{6-6t^2}{8t^3}$$

(e) Determine intervals of t-values for which the parametric curve is concave up and intervals for which it is concave down.

Concave Down: $\frac{6-6t^2}{8t^3} < 0$

$\Rightarrow t \in (-1,0) \cup (1,\infty)$

Concave Up: $\frac{6-6t^2}{8t^3} > 0$

$\Rightarrow t \in (-\infty,-1) \cup (0,1)$

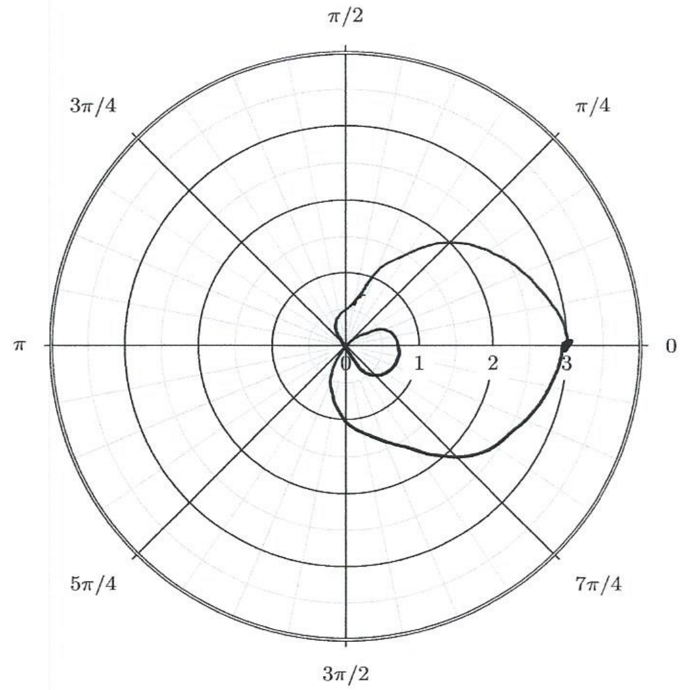
(f) Set up but **do not evaluate** an integral whose value gives the arc length of this curve from $t = 0$ to $t = 1$.

$$\int_0^1 \sqrt{(2t)^2 + (3-3t^2)^2}\, dt$$

(g) Set up but **do not evaluate** an integral whose values gives the area under this curve from $t = 0$ to $t = 1$.

$$\text{Area} = \int_0^1 (3t - t^3)(2t)\, dt$$

5. (10 points) Consider the polar curve defined by $r = 1 + 2\cos(\theta)$.



(a) Draw a clear sketch of the curve above. Look at the next page for intuition (nearly identical formula).

(b) For which values of θ does the curve cross itself?

$1 + 2\cos\theta$

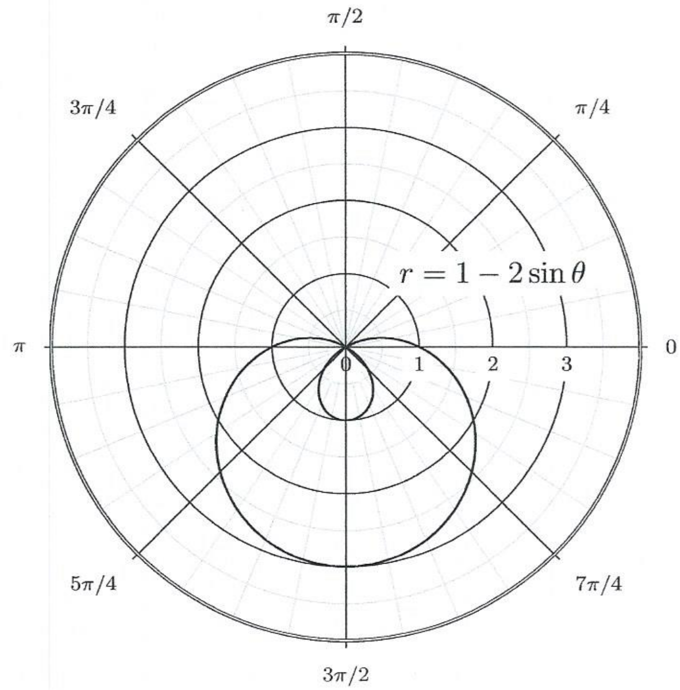
$\Rightarrow \cos\theta = -\frac{1}{2}$

$\theta = \frac{2\pi}{3}, \frac{4\pi}{3}$

(c) Write down but **do not evaluate** an integral whose value gives the arc length of the curve from $\theta = 0$ to $\theta = 2\pi$.

$$\int_0^{2\pi} \sqrt{r^2 + (r')^2}\, d\theta = \int_0^{2\pi} \sqrt{(1+2\cos\theta)^2 + (-2\sin\theta)^2}\, d\theta$$

6. (10 points) Consider the polar curve defined by $r = 1 - 2\sin(\theta)$. Write down but **do not evaluate** an integral whose value gives the area between the outer loop and the inner loop of the curve.



$1 - 2\sin\theta = 0$

$\Rightarrow \sin\theta = \frac{1}{2}$

$\Rightarrow \theta = \frac{\pi}{6}, \frac{5\pi}{6}$

$$\text{Area} = \underbrace{\int_0^{2\pi} \frac{1}{2}(1-2\sin\theta)^2 \, d\theta}_{} - 2\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \frac{1}{2}(1-2\sin\theta)^2 \, d\theta$$

This integral accounts for the area in the loop twice, so we must subtract the area in the loop twice.

7. (20 points) Determine whether each of the following sequences converges or diverges. If one converges, justify and find its limit. If it diverges, explain why.

(a) $a_n = \cos(n)$

Diverges since $\cos(n)$ oscillates irregularly.

(b) $a_n = \dfrac{\cos^2(n)}{n}$

$$0 \leq \left| \frac{\cos^2(n)}{n} \right| \leq \frac{1}{n}$$

Since $\lim_{n\to\infty} \frac{1}{n} = 0$, by the Squeeze Theorem, $\lim_{n\to\infty} a_n = 0$.

(c) $a_n = \sin\left(\frac{1}{n}\right)$

$$\lim_{n\to\infty} a_n = \sin\left(\lim_{n\to\infty} \frac{1}{n}\right) = \sin(0) = 0$$

$\sin(x)$ is continuous.

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(d) $a_n = \dfrac{6n^7 + 5n + 3}{4n^7 + 2n + 8}$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{6 + \overset{0}{\cancel{\frac{5}{n^6}}} + \overset{0}{\cancel{\frac{3}{n^7}}}}{4 + \underset{0}{\cancel{\frac{2}{n^6}}} + \underset{0}{\cancel{\frac{8}{n^7}}}} = \frac{6}{4} = \frac{3}{2}$$

(e) $a_n = \dfrac{n^2}{e^n - 1}$

$$f(x) = \frac{x^2}{e^x - 1}$$

$$\lim_{x\to\infty} f(x) \overset{L'H}{=} \lim_{x\to\infty} \frac{2x}{e^x} \overset{L'H}{=} \lim_{x\to\infty} \frac{2}{e^x} = 0, \text{ thus}$$

$$\lim_{n\to\infty} a_n = 0$$

(f) $a_n = \ln(n) - \ln(n^2 + 1)$

$$\lim_{n\to\infty} \ln\left(\frac{n}{n^2+1}\right) = \ln\left(\lim_{n\to\infty} \frac{n}{n^2+1}\right) = \ln(0) = -\infty,$$

thus it diverges.