Math 162: Calculus IIA

Second Midterm Exam ANSWERS

October 31, 2024

Integration by parts formula:

$$\int u\,dv = uv - \int v\,du$$

Trigonometric identities:

$$\cos^2(x) + \sin^2(x) = 1 \qquad \sec^2(x) - \tan^2(x) = 1 \qquad \sin(2x) = 2\sin(x)\cos(x)$$

$$\cos^2(x) = \frac{1+\cos(2x)}{2} \qquad \sin^2(x) = \frac{1-\cos(2x)}{2}$$

Derivatives of trig functions.

$$\frac{d\sin x}{dx} = \cos x \qquad \frac{d\tan x}{dx} = \sec^2 x \qquad \frac{d\sec x}{dx} = \sec x\tan x$$

$$\frac{d\cos x}{dx} = -\sin x \qquad \frac{d\cot x}{dx} = -\csc^2 x \qquad \frac{d\csc x}{dx} = -\csc x\cot x$$

Trigonometric substitution tricks for odd powers of secant and even powers of tangent:

$$u = \sec(\theta) + \tan(\theta) \qquad \sec(\theta)d\theta = \frac{du}{u}$$

$$\sec(\theta) = \frac{u^2+1}{2u} \qquad \tan(\theta) = \frac{u^2-1}{2u}$$

Area of surface of revolution in rectangular coordinates $y = f(x)$ with $0 \le a \le x \le b$:

- about the x-axis: $S = \int_a^b 2\pi|f(x)|\sqrt{1+[f'(x)]^2}\,dx$.
- about the y-axis: $S = 2\pi\int_a^b x\sqrt{1+[f'(x)]^2}\,dx$.

Polar coordinate formulas.

$$x = r\cos(\theta) \qquad r^2 = x^2 + y^2$$

$y = r\sin(\theta)$ $\tan(\theta) = y/x$

Note: $\theta = \arctan(y/x)$ when $x > 0$, and $\theta = \arctan(y/x) + \pi$ when $x < 0$.

Changing θ by any multiple of 2π does not change the location of the point.
Changing the sign of r is equivalent to adding π to θ, which is the same as moving the point to one in the opposite direction and the same distance from the origin.

Area in polar cooridinates for $r = f(\theta)$, with $\alpha \le \theta \le \beta$:

$$A = \int_{\alpha}^{\beta} \frac{r^2}{2}\, d\theta.$$

Arc length formulas:

- Rectangular coordinates, $y = f(x)$ with $a \le x \le b$:

$$L = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx.$$

- Polar coordinates, $r = f(\theta)$, $\alpha \le \theta \le \beta$:

$$L = \int_{\alpha}^{\beta} \sqrt{[f(\theta)]^2 + [f'(\theta)]^2}\, d\theta$$

- Parametric equations, $x = x(t), y = y(t)$ with $a \le t \le b$:

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt.$$

1. (20 points) The problem has four parts, each worth 5 points. Each depends on having correct answers to the previous parts. You should not expect to get credit for an answer based on incorrect information.

Consider the integral

$$\int \frac{x^3\,dx}{x^4 - 10x^2 + 9}.$$

(a) Write the denominator of the fraction as a product of linear factors.

Answer:

$$x^4 - 10x^2 + 9 = (x^2 - 1)(x^2 - 9) = (x-1)(x+1)(x-3)(x+3).$$

(b) Are there any values of x for which the integrand is not defined? If so, what are they and why?

Answer:

The denominator of the fraction is zero when x is ± 1 or ± 3. Since division by zero is not defined, the integrand is not defined at those values of x.

(c) Write the integrand as a sum of fractions with constant numerators, which you may denote by letters of the alphabet. YOU NEED NOT FIND THE VALUES OF THESE CONSTANTS.

Answer:

$$\frac{x^3}{x^4 - 10x^2 + 9} = \frac{A}{x-1} + \frac{B}{x+1} + \frac{C}{x-3} + \frac{D}{x+3}.$$

(d) Write the original integral in terms of the constants you named in part (c).

Answer:

$$\begin{aligned}\int \frac{x^3\,dx}{x^4 - 10x^2 + 9} &= \int \frac{A\,dx}{x-1} + \int \frac{B\,dx}{x+1} + \int \frac{C\,dx}{x-3} + \int \frac{D\,dx}{x+3} \\ &= A\ln|x-1| + B\ln|x+1| + C\ln|x-3| + D\ln|x+3| + c.\end{aligned}$$

2. (20 points) Find the length of the parametric curve

$$x = 2t - 1, \;\; y = t^2 + 6$$

where $0 \le t \le 3$.

Answer:

We have

$$\frac{dx}{dt} = 2, \qquad \frac{dy}{dt} = 2t,$$

and so

$$\begin{aligned} L &= \int_0^3 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt \\ &= \int_0^3 \sqrt{2^2 + (2t)^2}\, dt \\ &= \int_0^3 \sqrt{4 + 4t^2}\, dt \\ &= \int_0^3 \sqrt{4(1 + t^2)}\, dt \\ &= 2\int_0^3 \sqrt{1 + t^2}\, dt \end{aligned}$$

Now, making the substitution $t = \tan(\theta)$, $dt = \sec^2(\theta)\, d\theta$, we get

$$\begin{aligned} \int \sqrt{1+t^2}\, dt &= \int \sqrt{1 + \tan^2(\theta)}\; \sec^2(\theta)\, d\theta \\ &= \int \sec^3(\theta)\, d\theta \\ &= \frac{1}{2}\sec(\theta)\tan(\theta) + \frac{1}{2}\ln|\sec(\theta) + \tan(\theta)| + C \\ &= \frac{1}{2}\sqrt{1+t^2}t + \frac{1}{2}\ln|\sqrt{1+t^2} + t| + C \\ &\text{(using } \; t = \tan(\theta),\; \sec(\theta) = \sqrt{1 + \tan^2(\theta)} = \sqrt{1+t^2}). \end{aligned}$$

Therefore

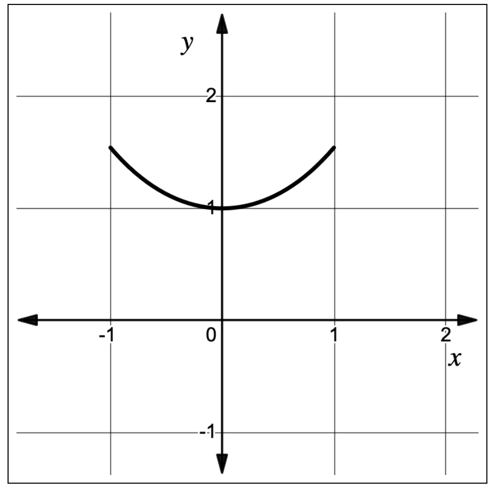
$$\begin{aligned} L &= 2\int_0^3 \sqrt{1+t^2}\, dt \\ &= 2\left[\frac{1}{2}t\sqrt{1+t^2} + \frac{1}{2}\ln(\sqrt{1+t^2} + t)\right]_{t=0}^3 \\ &= 3\sqrt{1+3^2} + \ln(\sqrt{1+3^2} + 3) \\ &= 3\sqrt{10} + \ln(\sqrt{10} + 3) \end{aligned}$$

3. (20 points)

The curve

$$y = \frac{e^x + e^{-x}}{2}, \quad -1 \le x \le 1,$$

is graphed to the right. Find the surface area of the solid of revolution obtained by rotating this curve about the x-axis.



Answer:

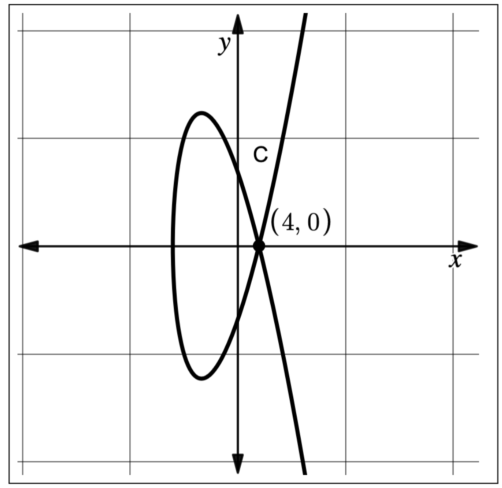
We have $\dfrac{dy}{dx} = \dfrac{e^x - e^{-x}}{2}$ so that

$$\begin{aligned}
1 + \left(\frac{dy}{dx}\right)^2 &= 1 + \frac{(e^x)^2 - 2e^x e^{-x} + (e^{-x})^2}{4} \\
&= \frac{4 + e^{2x} - 2 + e^{-2x}}{4} \\
&= \frac{e^{2x} + 2 + e^{-2x}}{4} \\
&= \left(\frac{e^x + e^{-x}}{2}\right)^2 = y^2.
\end{aligned}$$

Therefore, as we are rotating about the x-axis and both y, $\sqrt{1 + (dy/dx)^2}$ are even functions of x, we have

$$\begin{aligned}
S &= \int_{-1}^{1} 2\pi y \sqrt{1 + (dy/dx)^2}\, dx \\
&= 2\int_0^1 2\pi y \sqrt{y^2}\, dx = 4\pi \int_0^1 y^2\, dx = 4\pi \int_0^1 \frac{1}{4}\left(e^{2x} + 2 + e^{-2x}\right)\, dx \\
&= \pi \left[\frac{1}{2}e^{2x} + 2x - \frac{1}{2}e^{-2x}\right]_{x=0}^{1} \\
&= \pi \left(\left(\frac{1}{2}e^2 + 2 - \frac{1}{2}e^{-2}\right) - \left(\frac{1}{2}e^0 + 0 - \frac{1}{2}e^{-0}\right)\right) \\
&= \pi \left(\frac{1}{2}e^2 + 2 - \frac{1}{2}e^{-2}\right).
\end{aligned}$$

4. (20 points)



The curve C is defined by the parametric equation

$$x = t^2 - 12, \qquad y = t^3 - 16t.$$

Find the equation(s) of the tangent line(s) at the point $(4, 0)$. Note that C intersects itself at $(4, 0)$.

Answer:

We first find the slope dy/dx of a tangent line as a function of t. We have

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{3t^2 - 16}{2t}.$$

Next we find the time(s) t so that $(x(t), y(t)) = (4, 0)$. We have

$$t^2 - 12 = x(t) = 4 \implies t^2 = 16$$

and so $t = \pm 4$. Therefore, there are two tangent lines at $(4, 0)$; one tangent corresponds to $t = -4$, and the other corresponds to $t = 4$.

The slope of the tangent line when $t = -4$ is

$$\left.\frac{dy}{dx}\right|_{t=-4} = \frac{3(-4)^2 - 16}{2(-4)} = \frac{32}{-8} = -4.$$

So, when $t = -4$, the equation of the tangent line is

$$y - 0 = (-4)(x - 4).$$

When $t = 4$, we have

$$\left.\frac{dy}{dx}\right|_{t=4} = \frac{3(4)^2 - 16}{2(4)} = \frac{32}{8} = 4.$$

So, when $t = 4$, the equation of the tangent line is

$$y - 0 = 4(x - 4).$$

5. (20 points) Hippopede curves (pictured below) are a family of polar curves, defined by the equations

$$r(\theta) = 2\sqrt{ab - b^2 \sin^2(\theta)} \qquad 0 \le \theta \le 2\pi$$

Find the area of the Hippopede curve where $a = 2$ and $b = 1$.

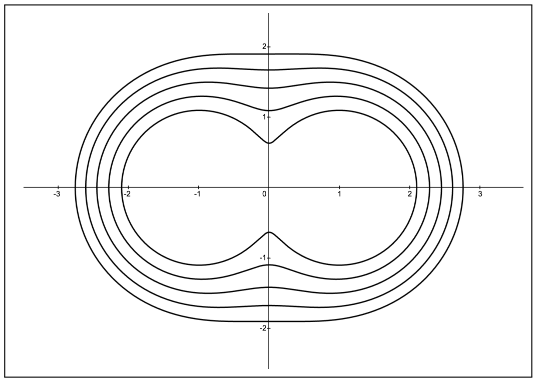


Figure 1: Five different Hippopede curves

Answer:

The formula for the area of a polar curve bounded between $\theta = \alpha$ and $\theta = \beta$ is given by

$$A = \frac{1}{2}\int_{\alpha}^{\beta} (r(\theta))^2 d\theta$$

In this case, $\alpha = 0$ and $\beta = 2\pi$. Then, since $a = 2$ and $b = 1$,

$$\frac{1}{2}\int_0^{2\pi} (r(\theta))^2 d\theta = \frac{1}{2}\int_0^{2\pi} 4(ab - b^2\sin^2(\theta))d\theta = \int_0^{2\pi} 2(2 - \sin^2(\theta))d\theta$$

Then, using $\sin^2\theta = \frac{1}{2}(1 - \cos(2\theta))$, this becomes

$$\int_0^{2\pi} 4 - (1 - \cos(2\theta))d\theta = \int_0^{2\pi} 3 - \cos(2\theta)d\theta$$

This is then

$$3\theta - \frac{1}{2}\sin(2\theta)\Big|_0^{2\pi} = 6\pi$$

Scratch paper

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