Math 162: Calculus IIA

Second Midterm Exam ANSWERS

November 14, 2023

Integration by parts formula:

$$\int u\,dv = uv - \int v\,du$$

Trigonometric identities:

$$\cos^2(x) + \sin^2(x) = 1 \qquad \sec^2(x) - \tan^2(x) = 1$$

$$\sin(2x) = 2\sin(x)\cos(x)$$

$$\cos^2(x) = \frac{1+\cos(2x)}{2} \qquad \sin^2(x) = \frac{1-\cos(2x)}{2}$$

Derivatives of trig functions.

$$\frac{d\sin x}{dx} = \cos x \qquad \frac{d\tan x}{dx} = \sec^2 x \qquad \frac{d\sec x}{dx} = \sec x\tan x$$

$$\frac{d\cos x}{dx} = -\sin x \qquad \frac{d\cot x}{dx} = -\csc^2 x \qquad \frac{d\csc x}{dx} = -\csc x\cot x$$

Trigonometric substitution (known in Doug's section as *the rabbit trick.*)) for odd powers of secant and even powers of tangent:

$$u = \sec(\theta) + \tan(\theta) \qquad \sec(\theta)d\theta = \frac{du}{u}$$

$$\sec(\theta) = \frac{u^2+1}{2u} \qquad \tan(\theta) = \frac{u^2-1}{2u}$$

Area of surface of revolution in rectangular coordinates, $y = f(x)$ with $a \le x \le b$

- about the x-axis: $$S = 2\pi\int_a^b f(x)\sqrt{1+f'(x)^2}\,dx$$
- about the y-axis: $$S = 2\pi\int_a^b x\sqrt{1+f'(x)^2}\,dx$$

More formulas for your enjoyment

Polar coordinates

$$r = \sqrt{x^2 + y^2}$$

$$\theta = \begin{cases} \arctan(y/x) & \text{for } x > 0 \\ \pi + \arctan(y/x) & \text{for } x < 0 \\ \pi/2 & \text{for } x = 0 \text{ and } y > 0 \\ 3\pi/2 & \text{for } x = 0 \text{ and } y < 0 \\ \text{undefined} & \text{for } (x, y) = (0, 0) \end{cases}$$

$$x = r\cos\theta \qquad y = r\sin\theta$$

Changing θ by any multiple of 2π does not change the location of the point. Changing the sign of r is equivalent to adding π to θ, which is the same as moving the point to one in the opposite direction and the same distance from the origin.

Area in polar coordinates for $r = f(\theta)$ with $\alpha \leq \theta \leq \beta$:

$$A = \int_\alpha^\beta \frac{r^2}{2}\, d\theta$$

Arc length formulas

- Rectangular coordinates, $y = f(x)$ with $a \leq x \leq b$:

$$S = \int_a^b \sqrt{1 + f'(x)^2}\, dx$$

- Polar coordinates, $r = f(\theta)$ with $\alpha \leq \theta \leq \beta$:

$$S = \int_\alpha^\beta \sqrt{r^2 + f'(\theta)^2}\, d\theta$$

- Parametric equations, $x = x(t)$ and $y = y(t)$ with $a \leq t \leq b$:

$$S = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

1. (20 points) Let $a \geq 3$. Determine whether the following improper integral converges. If it does, find its value in terms of a.

$$\int_a^\infty \frac{1}{x\,(\ln x)^2}\,dx$$

Answer:

Using the substitution $u = \ln x$, $du = \frac{1}{x}dx$, we have

$$\int \frac{1}{x\,(\ln x)^2}\,dx = \int \frac{1}{u^2}\,du = -\frac{1}{u} + C = -\frac{1}{\ln x} + C.$$

Using this, we can evaluate the improper integral as following limit:

$$\begin{aligned}\int_a^\infty \frac{1}{x\,(\ln x)^2}\,dx &= \lim_{N\to\infty} \int_a^N \frac{1}{x\,(\ln x)^2}\,dx \\ &= \lim_{N\to\infty} \left(-\frac{1}{\ln x}\Bigg|_{x=a}^{x=N} \right) \\ &= \lim_{N\to\infty} \left(-\frac{1}{\ln N} + \frac{1}{\ln a} \right) = \frac{1}{\ln a}.\end{aligned}$$

2. (20 points) A curve is described by the following parametric equation:

$$x = t^4 + 5, \quad y = F(t)$$

where $F(t)$ is defined by

$$F(t) = \int_0^t \sqrt{8u^3+1}\,du.$$

Find the length of the curve from $t = 0$ to $t = 1$.

Hint: You will have to use the fundamental theorem of calculus to find dy/dt. Do not attempt to find an antiderivative of $\sqrt{8u^3+1}$.

Answer:

We need to first find dx/dt and dy/dt. We have

$$\frac{dx}{dt} = 4t^3, \quad \frac{dy}{dt} = \frac{d}{dt}\left(\int_0^t \sqrt{8u^3+1}\,du \right) = \sqrt{8t^3+1},$$

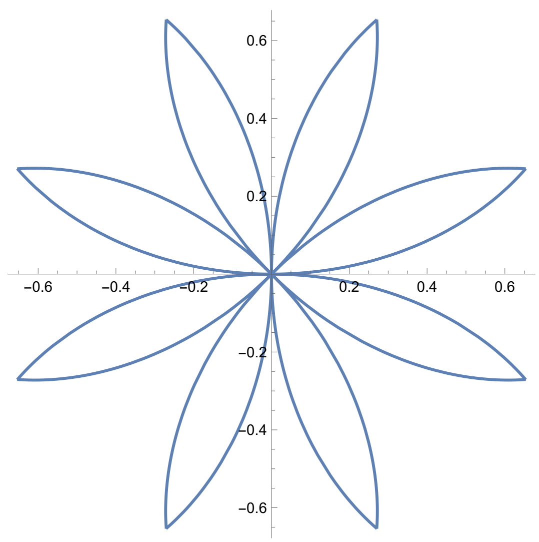
where we have used the fundamental theorem of calculus to find dy/dt. Therefore, when $t > 0$, we have

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = 16t^6 + 8t^3 + 1 = (4t^3+1)^2$$

and so

$$\begin{aligned} \text{length} &= \int_0^1 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt \\ &= \int_0^1 \sqrt{(4t^3+1)^2}\, dt \\ &= \int_0^1 4t^3 + 1\, dt \\ &= (t^4 + t)\Big]_{t=0}^{1} \\ &= 2. \end{aligned}$$

3. (20 points) Find the area of the region that lies inside the curves $r = \cos 2\theta$ and $r = \sin 2\theta$, which is shown below.



Answer:

The area inside the two curves divides into 8 equal regions. To find the total area, we can calculate the area of one region and then multiply it by 8. Therefore,

$$\text{Area} = 8\left(\int_0^{\pi/8} \frac{(\sin 2\theta)^2}{2} d\theta + \int_{\pi/8}^{\pi/4} \frac{(\cos 2\theta)^2}{2} d\theta\right)$$

$$= 2\int_0^{\pi/8} (1 - \cos 4\theta) d\theta + 2\int_{\pi/8}^{\pi/4} (1 + \cos 4\theta) d\theta$$

by the double angle formula

$$= \frac{1}{2}\int_0^{\pi/2}(1-\cos u)du + \frac{1}{2}\int_{\pi/2}^{\pi}(1+\cos u)du$$

where $u = 4\theta$, so $d\theta = du/4$

$$= \frac{1}{2}\int_0^{\pi} du - \frac{1}{2}\int_0^{\pi/2}\cos u\, du + \frac{1}{2}\int_{\pi/2}^{\pi}\cos u\, du$$

$$= \frac{\pi}{2} - \left.\frac{\sin u}{2}\right|_0^{\pi/2} + \left.\frac{\sin u}{2}\right|_{\pi/2}^{\pi}$$

$$= \frac{\pi}{2} - \frac{1-0}{2} + \frac{0-1}{2} = \frac{\pi}{2} - 1.$$

4. (20 points)

(a) 10 POINTS Compute the area of surface of revolution obtained by rotating the curve $y = \sqrt{4-x^2}$ around the x-axis.

Answer:

$$\begin{aligned} A &= 2\pi \int_{-2}^{2} y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx \\ &= 2\pi \int_{-2}^{2} \sqrt{4-x^2}\sqrt{1+\frac{x^2}{4-x^2}}\,dx \\ &= 2\pi \int_{-2}^{2} 2\,dx = 16\pi \end{aligned}$$

(b) 10 POINTS Do the same for the curve $y = 1-|x|$, $-1 \le x \le 1$.

Answer:

$$\begin{aligned} A &= 2\pi \int_{-1}^{0}(1+x)\sqrt{1+1}\,dx + 2\pi\int_0^1 (1-x)\sqrt{1+1}\,dx \\ &= 2\pi\sqrt{2}\left(\left[x+\frac{x^2}{2}\right]_{-1}^{0} + \left[x-\frac{x^2}{2}\right]_0^1\right) \\ &= 2\pi\sqrt{2}\left(\frac{1}{2}+\frac{1}{2}\right) = 2\pi\sqrt{2} \end{aligned}$$

5. (20 points) Determine if the following sequences are convergent or divergent and explain why. If it is convergent, give its limit.

(a) 10 POINTS

$$\left\{ \frac{n\cos(n)}{n^2+1} \;\middle|\; n \geq 0 \right\}$$

Answer:

Since for every $n \geq 1$ we have $-1 \leq \cos(n) \leq 1$ and hence

$$\frac{-n}{n^2+1} \leq \frac{n\cos(n)}{n^2+1} \leq \frac{n}{n^2+1}.$$

Consider

$$\lim_{n\to\infty} \frac{-n}{n^2+1} = \lim_{n\to\infty} \frac{n}{n^2+1} = 0.$$

By Squeeze Theorem, we prove $\lim\limits_{n\to\infty} \frac{n\cos(n)}{n^2+1} = 0$, and hence the sequence is convergent.

(b) 10 POINTS

$$\left\{ n^3 \sin\left(\frac{1}{n}\right) \;\middle|\; n \geq 1 \right\}$$

Answer:

Putting $x = 1/n$ we have

$$\lim_{n\to\infty} n^3 \sin\left(\frac{1}{n}\right) = \lim_{x\to 0^+} \frac{\sin x}{x^3} = \lim_{x\to 0^+} \frac{\cos x}{3x^2} = \infty.$$

Therefore, it is divergent.

Scratch paper

Scratch paper

Scratch paper

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