Math 162: Calculus IIA

Second Midterm Exam ANSWERS

November 11, 2021

Handy dandy formulas

Integration by parts formula:

$$\int u\,dv = uv - \int v\,du$$

Trigonometric identities:

$$\cos^2\theta + \sin^2\theta = 1 \qquad \sec^2\theta - \tan^2\theta = 1$$

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta \qquad \cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$

$$\cos^2\theta = \frac{1+\cos 2\theta}{2} \qquad \sin^2\theta = \frac{1-\cos 2\theta}{2}$$

Derivatives of trig functions.

$$\frac{d\sin x}{dx} = \cos x \qquad \frac{d\tan x}{dx} = \sec^2 x \qquad \frac{d\sec x}{dx} = \sec x\tan x$$

$$\frac{d\cos x}{dx} = -\sin x \qquad \frac{d\cot x}{dx} = -\csc^2 x \qquad \frac{d\csc x}{dx} = -\csc x\cot x$$

Trigonometric substitution for integrals of the form

$$\int \tan^m x\sec^n x\,dx \qquad \text{with } n > 0,$$

known in Doug's section as *the rabbit trick.*

$$u = \sec x + \tan x \qquad \sec x\,dx = \frac{du}{u}$$

$$\sec x = \frac{u^2+1}{2u} \qquad \tan x = \frac{u^2-1}{2u}$$

Area of surface of revolution in rectangular coordinates, $y = f(x)$ with $a \le x \le b$

- about the x-axis: $$S = 2\pi\int_a^b f(x)\sqrt{1+f'(x)^2}\,dx$$
- about the y-axis: $$S = 2\pi\int_a^b x\sqrt{1+f'(x)^2}\,dx$$

More formulas for your enjoyment

Polar coordinates

$$r = \sqrt{x^2 + y^2} \qquad \theta = \begin{cases} \arctan(y/x) & \text{for } x > 0 \\ \pi + \arctan(y/x) & \text{for } x < 0 \\ \pi/2 & \text{for } x = 0 \text{ and } y > 0 \\ 3\pi/2 & \text{for } x = 0 \text{ and } y < 0 \\ \text{undefined} & \text{for } (x, y) = (0, 0) \end{cases}$$

$$x = r\cos\theta \qquad y = r\sin\theta$$

Changing θ by any multiple of 2π does not change the location of the point. Changing the sign of r is equivalent to adding π to θ, which is the same as moving the point to one in the opposite direction and the same distance from the origin.

Area in polar coordinates for $r = f(\theta)$ with $\alpha \le \theta \le \beta$:

$$A = \int_\alpha^\beta \frac{r^2}{2}\, d\theta$$

Arc length formulas

- Rectangular coordinates, $y = f(x)$ with $a \le x \le b$:

$$S = \int_a^b \sqrt{1 + f'(x)^2}\, dx$$

- Polar coordinates, $r = f(\theta)$ with $\alpha \le \theta \le \beta$:

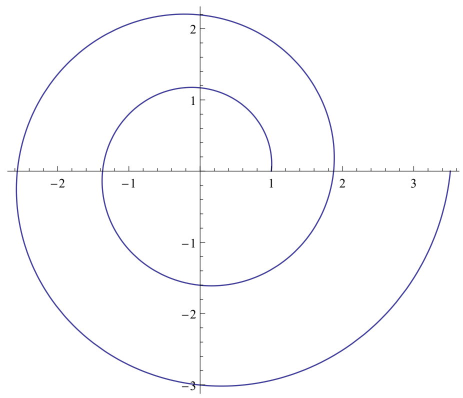
$$S = \int_\alpha^\beta \sqrt{r^2 + f'(\theta)^2}\, d\theta$$

- Parametric equations, $x = x(t)$ and $y = y(t)$ with $a \le t \le b$:

$$S = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

1. (20 points)

(a) (10 POINTS) Find the arc length of the polar curve $r = e^{k\theta}$ for $0 \le \theta \le 4\pi$ where $k = 1/10$, as shown below. You may express your answer in terms of e, π and $\sqrt{1.01}$.



Answer:

We have

$$r = e^{k\theta} \qquad \text{and} \qquad \frac{dr}{d\theta} = ke^{k\theta}.$$

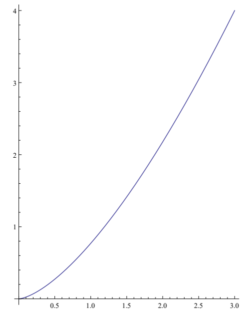
The polar arc length formula gives

$$\begin{aligned} S &= \int_0^{4\pi} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta \\ &= \int_0^{4\pi} \sqrt{e^{2k\theta} + (ke^{k\theta})^2}\, d\theta \\ &= \sqrt{1+k^2} \int_0^{4\pi} e^{k\theta}\, d\theta \\ &= \sqrt{1+k^2}\, \frac{e^{k\theta}}{k}\bigg|_0^{4\pi} \\ &= \sqrt{1+k^2}\left(\frac{e^{4\pi k} - 1}{k}\right) \\ &= 10\sqrt{1.01}\left(e^{4\pi/10} - 1\right) \end{aligned}$$

(b) (10 POINTS) Find the arc length of the parametric curve

$$x = 3t^2 \qquad\qquad y = 4t^3$$

for $0 \le t \le 1$, as shown below. HINT: Your answer must exceed the length of the straight line joining the two endpoints, which is 5.



Answer:

We have

$$\frac{dx}{dt} = 6t \qquad\qquad \frac{dy}{dt} = 12t^2$$

$$\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{36t^2 + 144t^4} = 6t\sqrt{1+4t^2}$$

Hence the arc length is

$$S = \int_0^1 6t\sqrt{1+4t^2}\,dt$$

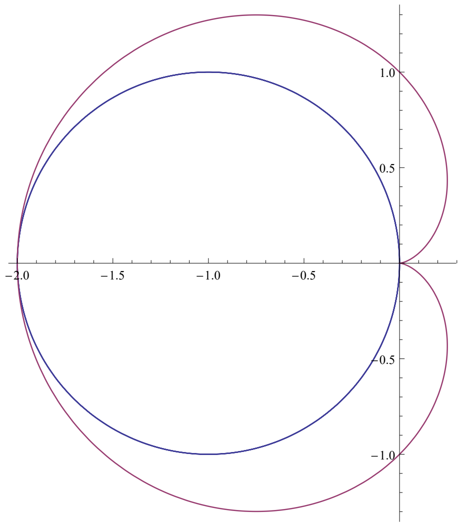
For this we use the substitution

$$u = 1 + 4t^2 \qquad\qquad du = 8t\,dt \qquad\qquad 6t\,dt = \frac{3}{4}du$$

In terms of u our integral is

$$S = \frac{3}{4}\int_1^5 \sqrt{u}\,du \;= \frac{3}{4}\,\frac{u^{3/2}}{3/2}\bigg|_1^5 = \frac{1}{2}(5^{3/2} - 1).$$

2. (20 points) Find the area of the cardioid defined in polar coordinates by $r = 1 - \cos\theta$ for $0 \le \theta \le 2\pi$, shown below. Write your answer in terms of π. Hint: The cardioid contains a circle of radius 1, also shown below, so its area must exceed π.



Answer:

The polar area formula gives

$$\begin{aligned} A &= \int_0^{2\pi} \frac{r^2}{2}\, d\theta = \frac{1}{2}\int_0^{2\pi} (1 - 2\cos\theta + \cos^2\theta)\, d\theta \\ &= \frac{1}{2}\int_0^{2\pi} (1 - 2\cos\theta + (1+\cos 2\theta)/2)\, d\theta \\ &= \frac{1}{2}\int_0^{2\pi} \frac{(3 - 4\cos\theta + \cos 2\theta)}{2}\, d\theta \\ &= \left(\frac{3\theta - 4\sin\theta + (\sin 2\theta)/2}{4}\right)\Bigg|_0^{2\pi} \\ &= 3\pi/2 \end{aligned}$$

3. (20 points) Determine if each of the following sequences is convergent or divergent, and find the limit if the sequence is convergent. For full credit, your work must support your answers.

(a) (10 POINTS)

$$\{a_n\}_{n=1}^{\infty} = \left\{\frac{n + \cos^2\left(\frac{1}{n}\right)}{n+1}\right\}_{n=1}^{\infty}$$

Answer:

The sequence $\{a_n\}$ converges to 1:

$$\begin{aligned}\lim_{n\to\infty} a_n &= \lim_{n\to\infty} \frac{n+\cos^2\left(\frac{1}{n}\right)}{n+1} \\ &= \lim_{n\to\infty} \frac{n}{n+1} + \frac{\cos^2\left(\frac{1}{n}\right)}{n+1} \\ &= \lim_{n\to\infty} \frac{n}{n+1} + \lim_{n\to\infty} \frac{\cos^2\left(\frac{1}{n}\right)}{n+1} \\ &= 1+0 \\ &= 1\end{aligned}$$

(b) (10 POINTS)

$$\{b_n\}_{n=1}^{\infty} = \left\{n^{-1}e^n\right\}_{n=1}^{\infty}$$

Answer:

The sequence $\{b_n\}$ diverges. Let $f(x) = x^{-1}e^x = \dfrac{e^x}{x}$. Then, by L'Hôpital's rule,

$$\lim_{x\to\infty} \frac{e^x}{x} = \lim_{x\to\infty} \frac{e^x}{1} = +\infty.$$

Hence, $\lim_{n\to\infty} b_n = +\infty$, so the sequence diverges.

4. (20 points)

Compute the surface area of the solid formed by revolving $y = x^3$ for $0 \le x < 1$ about the x-axis.

Answer:

$\frac{dy}{dx} = 3x^2$, so $(\frac{dy}{dx})^2 = 9x^4$. So the surface area formula is

$$2\pi \int_0^1 x^3\sqrt{9x^4+1}dx.$$

$u = 9x^4 + 1$, so $du = 36x^3dx$, i.e. $du/36 = x^3dx$. The bounds are transformed to $u = 9(0)^4 + 1 = 1$ to $u = 9(1)^4 + 1 = 10$.

Substituting this, we get that the integral is

$$
\begin{aligned}
&2\pi \int_1^{10} \frac{1}{36}\sqrt{u}du \\
=&\frac{\pi}{18}\left[\frac{2}{3}u^{3/2}\right]_1^{10} \\
=&\frac{\pi}{27}(10^{3/2}-1).
\end{aligned}
$$

5. (20 points) Compute the following indefinite integral:

$$\int \frac{dx}{(x-2)^2(x+1)}$$

Answer:

We will use partial fractions:

$$
\begin{aligned}
\frac{1}{(x-2)^2(x+1)} &= \frac{A}{x-2} + \frac{B}{(x-2)^2} + \frac{C}{x+1} \\
1 &= A(x-2)(x+1) + B(x+1) + C(x-2)^2
\end{aligned}
$$

Using the Heaviside Method, we set x equal to the roots:

$x = 2 : 1 = B(2+1) \to B = \frac{1}{3}$.
$x = -1 : 1 = C(-1-2)^2 \to C = \frac{1}{9}$.

To find A, we need to match the coefficients of x monomials: $1 = A(x^2 - x - 2) + \frac{1}{3}(x+1) + \frac{1}{9}(x^2 - 4x + 4)$.

The coefficient of x^2 on the left side is 0, and on the right side it is $A + \frac{1}{9}$. So $0 = A + \frac{1}{9} \to A = -\frac{1}{9}$.

Therefore,

$$
\begin{aligned}
&\int \frac{dx}{(x-2)^2(x+1)} \\
=&\int \frac{-dx}{9(x-2)} + \frac{dx}{3(x-2)^2} + \frac{dx}{9(x+1)}
\end{aligned}
$$

$$= -\frac{1}{9}\ln|x-2| - \frac{1}{3(x-2)} + \frac{1}{9}\ln|x+1| + C.$$

Scratch paper

More scratch paper