Math 162: Calculus IIA

Second Midterm Exam, Morning Edition ANSWERS

November 7, 2020

Trig formulas:

- $\cos^2(x) + \sin^2(x) = 1$
- $\sec^2(x) - \tan^2(x) = 1$
- $\sin(2x) = 2\sin(x)\cos(x)$
- $\cos^2(x) = \dfrac{1+\cos(2x)}{2}$
- $\sin^2(x) = \dfrac{1-\cos(2x)}{2}$

Polar coordinate formulas:

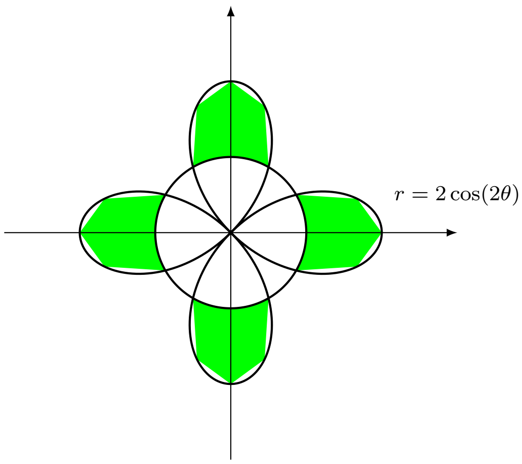
- Area:
$$\frac{1}{2}\int r^2 d\theta$$
- Arc length:
$$\int \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta$$

Parametric equation formulas:

- Newton's notation: $\dot{x} = dx/dt \qquad \dot{y} = dy/dt$
- Slope of tangent line: $dy/dx = \dot{y}/\dot{x}$.
- Second derivative
$$\frac{d^2y}{dx^2} = \frac{d(\dot{y}/\dot{x})/dt}{\dot{x}}.$$
Curve is concave up/down when this is positive/negative.
- Arc length:
$$\int \sqrt{\dot{x}^2 + \dot{y}^2} dt$$

1. (25 points)

Compute the area inside the polar curve $r = 2\cos(2\theta)$, a four leafed rose, and outside $r = 1$, a circle.



Answer:

Solving $\pm 1 = 2\cos(2\theta)$ for θ in $[0, 2\pi]$, we get

$$\theta = \left\{ \frac{\pi}{6}, \frac{\pi}{3}, \frac{2\pi}{3}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{4\pi}{3}, \frac{5\pi}{3}, \frac{11\pi}{6} \right\}.$$

We can find this area in one of 2 ways:

(1): Using the area formula, we compute

$$\int_0^{\pi/6} \frac{1}{2}\left[(2\cos(2\theta))^2 - (1)^2\right] d\theta + \int_{\pi/3}^{2\pi/3} \frac{1}{2}\left[(2\cos(2\theta))^2 - (1)^2\right] d\theta$$
$$+ \int_{5\pi/6}^{7\pi/6} \frac{1}{2}\left[(2\cos(2\theta))^2 - (1)^2\right] d\theta + \int_{4\pi/3}^{5\pi/3} \frac{1}{2}\left[(2\cos(2\theta))^2 - (1)^2\right] d\theta$$
$$+ \int_{11\pi/6}^{2\pi} \frac{1}{2}\left[(2\cos(2\theta))^2 - (1)^2\right] d\theta$$

Using a double angle formula and simplifying this equals

$$\int_0^{\pi/6} \left[\cos(4\theta) + \frac{1}{2}\right] d\theta + \int_{\pi/3}^{2\pi/3} \left[\cos(4\theta) + \frac{1}{2}\right] d\theta$$
$$+ \int_{5\pi/6}^{7\pi/6} \left[\cos(4\theta) + \frac{1}{2}\right] d\theta + \int_{4\pi/3}^{5\pi/3} \left[\cos(4\theta) + \frac{1}{2}\right] d\theta$$
$$+ \int_{11\pi/6}^{2\pi} \left[\cos(4\theta) + \frac{1}{2}\right] d\theta$$

$$
\begin{aligned}
&= \frac{\pi}{12} + \frac{\sqrt{3}}{8} + \frac{\pi}{6} + \frac{\sqrt{3}}{4} + \frac{\pi}{6} + \frac{\sqrt{3}}{4} + \frac{\pi}{6} + \frac{\sqrt{3}}{4} + \frac{\pi}{12} + \frac{\sqrt{3}}{8} \\
&= \frac{2\pi}{3} + \sqrt{3}.
\end{aligned}
$$

(2): Alternatively, you could have observed that this is 8 symmetric regions, and computed just one.

$$
\begin{aligned}
& 8 \int_0^{\pi/6} \frac{1}{2} \left[(2\cos(2\theta))^2 - (1)^2 \right] d\theta \\
&= 8 \cdot \left(\frac{\pi}{12} + \frac{\sqrt{3}}{8} \right) \\
&= \frac{2\pi}{3} + \sqrt{3}.
\end{aligned}
$$

2. (25 points)

Let C be the upper half of the circle centered at the origin with radius 3. The arc on C between $(0,3)$ and $(\sqrt{5},2)$ is rotated about the x-axis to produce a surface S.

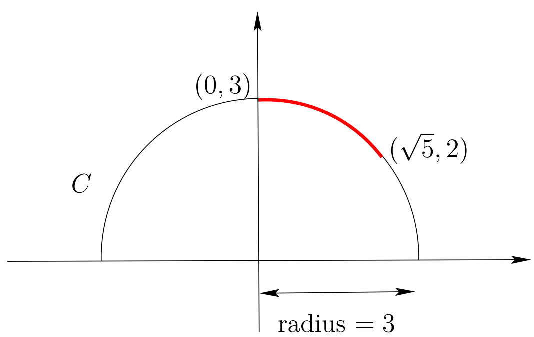


Figure 1:

(a) (15 points) Use

$$ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

to find the surface area.

Answer:

The formula for C is $y = \sqrt{9 - x^2}$. Then

$$\begin{aligned}
S &= \int_0^{\sqrt{5}} 2\pi y \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx \\
&= \int_0^{\sqrt{5}} 2\pi \sqrt{9-x^2} \sqrt{1+\left(-\frac{x}{\sqrt{9-x^2}}\right)^2}\, dx \\
&= \int_0^{\sqrt{5}} 2\pi \sqrt{9-x^2} \sqrt{\frac{9}{9-x^2}}\, dx \\
&= \int_0^{\sqrt{5}} 6\pi\, dx \\
&= 6\sqrt{5}\pi
\end{aligned}$$

(b) (10 points) Consider the same surface as in part (a). This time, use

$$ds = \sqrt{1+\left(\frac{dx}{dy}\right)^2}\, dy$$

to find the surface area.

Answer:

Since the arc lies on the right half of C, we use the formula $x = \sqrt{9-y^2}$, so that

$$\begin{aligned}
S &= \int_2^3 2\pi y \sqrt{1+\left(\frac{dx}{dy}\right)^2}\, dy \\
&= \int_2^3 2\pi y \sqrt{1+\left(-\frac{y}{\sqrt{9-y^2}}\right)^2}\, dy \\
&= \int_2^3 2\pi y \sqrt{\frac{9}{9-y^2}}\, dy \\
&= \int_2^3 6\pi y \,\frac{1}{\sqrt{9-y^2}}\, dy
\end{aligned}$$

Make the substitution $u = 9 - y^2$, so that

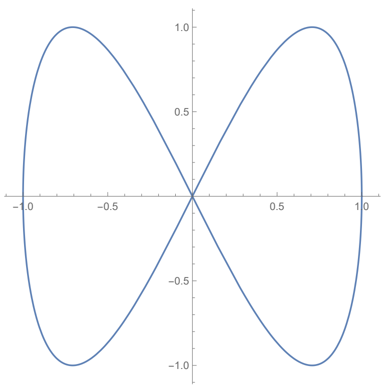
$$-du = 2y\, dy, \quad \text{when} \quad y = 2, u = 5, \quad \text{when} \quad y = 3, u = 0.$$

The integral becomes

$$\begin{aligned} S &= -\int_5^0 3\pi \frac{1}{\sqrt{u}}\, du \\ &= \int_0^5 3\pi \frac{1}{\sqrt{u}}\, du \\ &= 6\pi\sqrt{u}\ \Big|_0^5 \\ &= 6\sqrt{5}\pi \end{aligned}$$

3. (25 points) Consider the parametric curve

$$x = \cos(t), \qquad y = \sin(2t), \qquad 0 \le t \le 2\pi$$



(a) (9 points) At what points is the tangent horizontal or vertical?

Answer:

We have

$$\frac{dy}{dx} = -\frac{2\cos 2t}{\sin t}$$

The tangent line is horizontal when this derivative is 0, namely when $t = \dfrac{\pi}{4}, \dfrac{3\pi}{4}, \dfrac{5\pi}{4}, \dfrac{7\pi}{4}$. The corresponding four Cartesian points are $(\pm\sqrt{2}/2, \pm 1)$.

The tangent line is vertical when the derivative is undefined, namely at $t = 0$ and $t = \pi$. The corresponding two Cartesian points are $(\pm 1, 0)$.

(b) (8 points) The curve passes through the origin twice. What are the slopes of the two tangent lines to the curve at the origin?

Answer:

The curve passes through the origin at $t = \frac{\pi}{2}$ and $t = \frac{3\pi}{2}$. The two slopes are 2 and -2, respectively.

(c) (8 points) Find the equation of the form $y = mx + b$ for the tangent at $t = \frac{\pi}{6}$.

Answer:

At $t = \pi/6$ we have $x = y = \sqrt{3}/2$ and $dy/dx = -2$, so the equation for the tangent line is

$$\begin{aligned}
\frac{y - \sqrt{3}/2}{x - \sqrt{3}/2} &= -2 \\
y - \sqrt{3}/2 &= -2\left(x - \sqrt{3}/2\right) = -2x + \sqrt{3} \\
y &= -2x + \frac{3\sqrt{3}}{2}.
\end{aligned}$$

4. (25 points)

Consider the logarithmic spiral $r = e^{\theta}$, $\theta \in [0, \infty)$, which can be defined parametrically by $x = e^t \cos t$ and $y = e^t \sin t$ with $t = \theta$.

(a) (13 points) Calculate the arc-length of the logarithmic spiral for $0 \le \theta \le b$.

Answer:

For the arc length we have

$$\begin{aligned}
\frac{dx}{dt} &= e^t(\cos t - \sin t) \\
\frac{dy}{dt} &= e^t(\sin t + \cos t) \\
\left(\frac{ds}{dt}\right)^2 &= \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 \\
&= e^{2t}\left((\cos t - \sin t)^2 + (\sin t + \cos t)^2\right) \\
&= e^{2t}\left((\cos^2 - 2\cos t \sin t + \sin^2 t) + (\cos^2 + 2\cos t \sin t + \sin^2 t)\right) \\
&= 2e^{2t} \\
\frac{ds}{dt} &= e^t\sqrt{2},
\end{aligned}$$

so

$$s = \sqrt{2}\int_0^b e^t dt = \sqrt{2}\, e^t\Big|_0^b = \sqrt{2}(e^b - 1).$$

(b) Calculate the area between the curve and the x-axis for $\theta \in [0, \pi]$.

Answer:

Using the area formula for polar coordinates, we have

$$A \;=\; \frac{1}{2}\int_0^{\pi} r^2 d\theta = \frac{1}{2}\int_0^{\pi} e^{2\theta} d\theta = \frac{1}{2}\,\frac{e^{2\theta}}{2}\bigg|_0^{\pi} = \frac{e^{2\pi}-1}{4}.$$