Math 162: Calculus IIA

Second Midterm Exam ANSWERS

November 18, 2016

1. (20 points)

The following are convergent series. Find the sum of each of them. **You don't need to justify the convergence of the series, but you should clearly show how you got the answer.**

(a)

$$\sum_{n=1}^{\infty} \frac{3}{4n^2+4n-3} = \sum_{n=1}^{\infty} \frac{3}{(2n+3)(2n-1)}$$

Answer:

(b)

$$\sum_{n=2}^{\infty} (-2)^{n+1} 3^{-n}$$

Answer:

(a) Since

$$\frac{1}{4n^2+4n-3} = \frac{1}{(2n-1)(2n+3)} = \frac{1}{4}\left(\frac{1}{2n-1} - \frac{1}{2n+3}\right),$$

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{3}{4n^2+4n-3} &= \frac{3}{4}\sum_{n=1}^{\infty}\left(\frac{1}{2n-1} - \frac{1}{2n+3}\right) \\ &= \frac{3}{4}\left[\left(1-\frac{1}{5}\right) + \left(\frac{1}{3}-\frac{1}{7}\right) + \left(\frac{1}{5}-\frac{1}{9}\right) + \cdots\right] \\ &= \frac{3}{4}\left(1+\frac{1}{3}\right) = 1.\end{aligned}$$

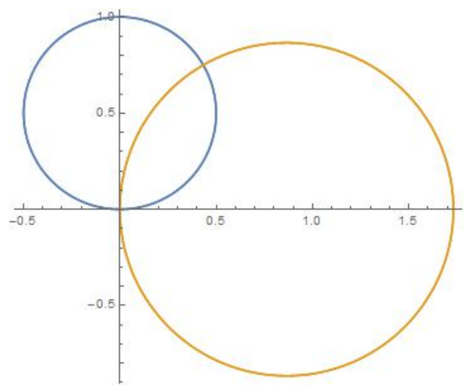
(b) Since

$$(-2)^{n+1}3^{-n} = (-2)\cdot\left(-\frac{2}{3}\right)^n,$$

$$\sum_{n=2}^{\infty}(-2)^{n+1}3^{-n} = (-2)\sum_{n=2}^{\infty}\left(-\frac{2}{3}\right)^n = (-2)\frac{\left(-\frac{2}{3}\right)^2}{1+\frac{2}{3}} = -\frac{8}{15}.$$

2. (20 points)

(a) Find the area of the region both inside the circle $r = \sin\theta$ and outside the circle $r = \sqrt{3}\cos\theta$ (both equations are in polar coordinates). The two circles are shown below. THEY INTERSECT AT THE ORIGIN AND THE POLAR POINT $(\theta, r) = (\pi/3, \sqrt{3}/2)$.



Answer:

(b) Compute the equation (in Cartesian coordinates x, y) of the tangent line to the circle $r = \sin\theta$ at the points where it intersects the circle $r = \sqrt{3}\cos\theta$

Answer:

(a) Find the area of the region inside the first circle and outside the second by integrating:

$$\int_{\pi/3}^{\pi} \tfrac{1}{2}r^2\,d\theta = \tfrac{1}{2}\int_{\pi/3}^{\pi}\sin^2\theta\,d\theta = \tfrac{1}{4}\int_{\pi/3}^{\pi}\big(1-\cos 2\theta\big)\,d\theta = \frac{\pi}{6}+\frac{\sqrt{3}}{16}$$

and subtracting:

$$\int_{\pi/3}^{\pi/2} \tfrac{1}{2}r^2\,d\theta = \tfrac{1}{2}\int_{\pi/3}^{\pi/2}3\cos^2\theta\,d\theta = \tfrac{3}{4}\int_{\pi/3}^{\pi/2}\big(1+\cos 2\theta\big)\,d\theta = \frac{\pi}{8}-\frac{3\sqrt{3}}{16}$$

So the area of the region is $\dfrac{\pi}{24}+\dfrac{\sqrt{3}}{4} \approx 0.563912.$

(b) Convert the curve to Cartesian coordinates:

$$x = r\cos\theta = \sin\theta\cos\theta = \tfrac{1}{2}\sin 2\theta$$
$$y = r\sin\theta = \sin^2\theta = \frac{1-\cos 2\theta}{2}$$

Thus:

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{\sin 2\theta}{\cos 2\theta} = \tan(2\theta)$$

So at the points of intersection $\theta = 0$ and $\theta = \pi/3$:

$$\frac{dy}{dx} = \tan(0) = 0 \qquad\qquad \frac{dy}{dx} = \tan(2\pi/3) = -\sqrt{3}$$

Since $(r, \theta) = (\sqrt{3}/2, \pi/2)$ corresponds to $(x, y) = (\sqrt{3}/4, 3/4)$ (scale the 1-2-$\sqrt{3}$ triangle by $\sqrt{3}/4$), the equations of the tangents at those points are:

$$y = 0 \qquad\qquad y - \frac{\sqrt{3}}{2} = -\sqrt{3}\big(x - \tfrac{\sqrt{3}}{4}\big)$$

3. (20 points)

Determine whether the following series converge or diverge. Justify your answers, making sure to name the convergence test(s) that you are using.

(a)

$$\frac{1}{5} + \frac{1}{8} + \frac{1}{11} + \frac{1}{14} + \frac{1}{17} + \frac{1}{20} + \cdots$$

Answer:

(b)

$$\sum_{n=2}^{\infty} \frac{2}{n(\ln(n))^2 + 1}$$

.

Answer:

(a) First write this as a series (there are many choices for this). One choice is: $\sum_{n=0}^{\infty} \frac{1}{3n+5}$. Note that the function $f(x) = 1/(3x+5)$ is positive, decreasing (various ways for showing this), and continuous on $[1, \infty)$. Now compute $\int_1^{\infty} f(x)\,dx = +\infty$, and so the series diverges by the integral test.

(b) Set $f(x) = 1/(x(\ln(x)^2 + 1)$. Note this is continuous on $[2, \infty)$, decreasing (various methods to show), and positive. Now use the integral test:

$$\int_2^\infty \frac{1}{x\ln(x)^2 + 1}\,dx \le \int_2^\infty \frac{1}{x\ln(x)^2}\,dx.$$

Use u-sub with $u = \ln(x)$ to show

$$\int_2^\infty \frac{1}{x\ln(x)^2}\,dx = \int_{\ln(2)}^\infty \frac{1}{u^2}\,du = 1/\ln(2).$$

Thus, the series converges by the integral test.

4. (20 points)

(a) Compute the area of surface of revolution obtained by rotating the curve $y = x^3$, for $0 \le x \le 1$, about the x-axis.

Answer:

(b) Do the same for the curve $y = |x|$, for $-1 \le x \le 1$.

Answer:

Solution:

(a)

$$\begin{aligned} A &= 2\pi \int_0^1 y\sqrt{1 + (dy/dx)^2}\,dx \\ &= 2\pi \int_0^1 x^3\sqrt{1 + 9x^4}\,dx \qquad\qquad (u = 1 + 9x^4,\ du = 36x^3\,dx) \\ &= \frac{2\pi}{36}\int_1^{10} \sqrt{u}\,du \\ &= \frac{2\pi}{36}\cdot\frac{2}{3}u^{3/2}\Big|_1^{10} \ = \ \frac{\pi}{27}\left(10^{3/2} - 1\right) \ = \ \frac{\pi}{27}\left(10\sqrt{10} - 1\right) \end{aligned}$$

(b) We have

$$\frac{dy}{dx} = -1\text{for } x < 01\text{for } x > 0$$

so

$$\begin{aligned} A &= 2\pi \int_{-1}^{0} (-x)\sqrt{1+(-1)^2}\,dx + 2\pi \int_{0}^{1} x\sqrt{1+1^2}\,dx \\ &= 2\pi\sqrt{2}\left(\left[-\frac{x^2}{2}\right]_{-1}^{0} + \left[\frac{x^2}{2}\right]_{0}^{1}\right) \\ &= 2\pi\sqrt{2}\left(\frac{1}{2}+\frac{1}{2}\right) = 2\pi\sqrt{2} \end{aligned}$$

5. (20 points)

Find the arc-length of the parametric curve

$$x = \cos t - \cos 5t\,,\ y = \sin t - \sin 5t\,,\ 0 \le t \le 2\pi\,.$$

by doing it for $0 \le t \le \pi/2$ and multiplying your answer by 4.

You may want to use the trig identities $\cos(\alpha - \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$ and $\sin^2\theta = (1 - \cos 2\theta)/2$.

PROFESSORIAL SCREWUP NOTICE: The problem as stated above has a typo that makes it unreasonably difficult. In consideration for this error, anyone who atttempted it will be given full credit. The intended problem is stated below, and the picture matches it.

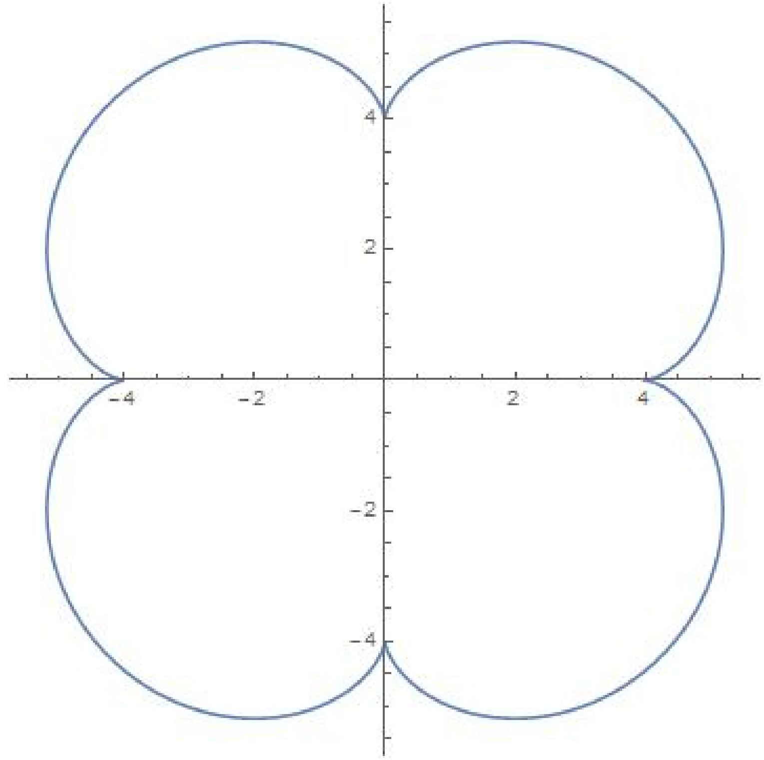
Find the arc-length of the parametric curve

$$x = 5\cos t - \cos 5t\,,\ y = 5\sin t - \sin 5t\,,\ 0 \le t \le 2\pi\,.$$

by doing it for $0 \le t \le \pi/2$ and multiplying your answer by 4.

You may want to use the trig identities $\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$ and $\sin^2\theta = (1 - \cos 2\theta)/2$.

The curve for $0 \le t \le 2\pi$ is pictured below.


4
2
-4
-2
2
4
-2
-4

Answer:

We have

$$dx/dt = -5(\sin t - \sin 5t) \qquad \text{and} \qquad dy/dt = 5(\cos t - \cos 5t).$$

Therefore

$$\begin{aligned}
(ds/dt)^2 &= (dx/dt)^2 + (dy/dt)^2 \\
&= 25(\sin t - \sin 5t)^2 + 25(\cos t - \cos 5t)^2 \\
&= 25(\sin^2 t - 2\sin t \sin 5t + \sin^2 5t + \cos^2 t - 2\cos t \cos 5t + \cos^2 5t) \\
&= 25(2 - 2\cos 4t) = 50(1 - \cos 4t) \\
&\qquad \text{since } \cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta \\
&= 100\left(\frac{1 - \cos 4t}{2}\right) \\
&= 100\sin^2 2t,
\end{aligned}$$

so

$$\frac{ds}{dt} = 10|\sin 2t|.$$

By the arc length formula, we have

$$\begin{aligned}
L &= 4\int_0^{\pi/2} ds = 40\int_0^{\pi/2} |\sin 2t|\, dt \\
&= 20\int_0^{\pi} \sin u\, du, \qquad \text{where } u = 2t \\
&= -20\cos u\Big|_0^{\pi} \\
&= 40.
\end{aligned}$$