Math 162: Calculus IIA

Second Midterm Exam ANSWERS

November 19, 2015

1. (20 points)

The following are convergent series. Find the sum of the series (You don't need to justify the convergence of the series, but you should clearly show how you got the answer).

(a)

$$\sum_{n=5}^{\infty} \frac{4}{n^2 - 2n}$$

Answer:

(b)

$$\sum_{n=1}^{\infty} \frac{(-1)^n + 1^n}{3^n}$$

Answer:

(a) From

$$\frac{4}{n^2 - 2n} = \frac{4}{n(n-2)} = \frac{2}{n-2} - \frac{2}{n},$$

$$\begin{aligned}\sum_{n=5}^{\infty} \frac{4}{n^2 - 2n} &= \left(\frac{2}{3} - \frac{2}{5}\right) + \left(\frac{2}{4} - \frac{2}{6}\right) + \left(\frac{2}{5} - \frac{2}{7}\right) + \left(\frac{2}{6} - \frac{2}{8}\right) + ... \\ &= \frac{2}{3} + \frac{2}{4} = \frac{7}{6}.\end{aligned}$$

(b)

$$\frac{(-1)^n + 1^n}{3^n} = \left(-\frac{1}{3}\right)^n + \left(\frac{1}{3}\right)^n.$$

From

$$\sum_{n=1}^{\infty} \left(-\frac{1}{3}\right)^n = -\frac{1}{3} \sum_{n=1}^{\infty} \left(-\frac{1}{3}\right)^{(n-1)} = -\frac{1}{3}\frac{1}{1+\frac{1}{3}} = -\frac{1}{4}$$

and

$$\sum_{n=1}^{\infty}\left(\frac{1}{3}\right)^n = \frac{1}{3}\sum_{n=1}^{\infty}\left(\frac{1}{3}\right)^{(n-1)} = \frac{1}{3}\frac{1}{1-\frac{1}{3}} = \frac{1}{2},$$

$$\sum_{n=1}^{\infty}\frac{(-1)^n + 1^n}{3^n} = -\frac{1}{4} + \frac{1}{2} = \frac{1}{4}.$$

2. (20 points) (a) Find the area of the region that is inside the circle given by $r = 2\sin\theta$ and outside the circle given by $r = 1$ (both equations are in polar coordinates).

Answer:

(b) Compute the equation of the tangent line to the circle $r = 2\sin\theta$ at the point of intersection with the circle $r = 1$ in the first quadrant.

Answer:

(a) First we find the values of θ at which the two circles intersect. Setting $2\sin\theta = 1$ we find that $\sin\theta = \frac{1}{2}$ and so $\theta = \frac{\pi}{6}$ or $\pi - \frac{\pi}{6}$. So using the area in polar coordinates and double angle formula for $\sin^2$

$$A = \int_{\frac{\pi}{6}}^{\pi-\frac{\pi}{6}} \frac{1}{2}((2\sin\theta)^2 - 1)\,d\theta = \int_{\frac{\pi}{6}}^{\pi-\frac{\pi}{6}} (1 - \cos(2\theta) - \frac{1}{2})\,d\theta$$
$$= (\frac{\theta}{2} - \frac{\sin(2\theta)}{2})\,|_{\frac{\pi}{6}}^{\pi-\frac{\pi}{6}} = \frac{\pi}{3} - \frac{\sin(2\pi - \frac{\pi}{3})}{2} + \frac{1}{2}\sin\frac{\pi}{3} = \frac{\pi}{3} + \frac{\sqrt{3}}{2}$$

This example could have also been computed with symmetry by integrating

$$A = 2\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{1}{2}((2\sin\theta)^2 - 1)\,d\theta.$$

(b) First, note that this point of intersection was computed in part a) with $\theta = \frac{\pi}{6}$. Then converting to Cartesian coordinates we get parametric equation

$$x = r\cos\theta = 2\sin\theta\cos\theta = \sin(2\theta)\ y = r\sin\theta = 2\sin^2\theta$$

Thus

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{4\sin\theta\cos\theta}{2\cos(2\theta)} = \frac{\sin(2\theta)}{\cos(2\theta)} = \tan(2\theta)$$

Hence at the point of intersection $\frac{dy}{dx} = \tan(2\frac{\pi}{6}) = \tan(\pi/3) = \sqrt{3}$.

The equation of the line is $y = mx + b$, where since it is tangent, we know the slope $m = \frac{dy}{dx} = \sqrt{3}$ and the line passes through the point that has polar coordinates $(1, \frac{\pi}{6}$. The point of tangency in standard coordinates is $(x, y) = (r\cos\theta, r\sin\theta) = (1\frac{\sqrt{3}}{2}, \frac{1}{2})$, hence

$\frac{1}{2} = \sqrt{3}\cdot\frac{\sqrt{3}}{2} + b$ and $b = -1$. Equation of the tangent line is then $y = \sqrt{3}x - 1$

3. (20 points) Determine whether the following series converge or diverge. Justify your answers, making sure to name the convergence test(s) that you are using.

(a)

$$\sum_{n=1}^{\infty} \frac{n^3 \sin^2(1/n)}{n+1} = \frac{\sin^2(1)}{2} + \frac{8\sin^2(1/2)}{3} + \frac{27\sin^2(1/3)}{4} + \cdots$$

Answer:

(b)

$$\sum_{n=1}^{\infty} ne^{-n^{3/2}} = e^{-1} + 2e^{-\sqrt{8}} + 3e^{-\sqrt{27}} + \cdots$$

Answer:

(a) Notice that

$$\lim_{n\to\infty} \frac{n^3\sin^2(1/n)}{n+1} = \left(\lim_{x\to 0}\left(\frac{\sin x}{x}\right)^2\right)\left(\lim_{n\to\infty}\left(\frac{n}{n+1}\right)\right) = 1,$$

so that $\lim_{n\to\infty} \frac{n^3\sin^2(1/n)}{n+1} = 1 \neq 0$, and by the divergence test we conclude that $\sum_{n=1}^{\infty} \frac{n^3\sin^2(1/n)}{n+1}$ diverges.

(b) We use the integral test: $f(x) = xe^{-x^{3/2}}$ is continuous, positive and decreasing for $x > 1$. Note that

$$xe^{-x^{3/2}} \le xe^{-x}, \quad \text{for all } x \ge 1.$$

By comparison we have

$$\int_1^{\infty} xe^{-x^{3/2}}dx \le \int_1^{\infty} xe^{-x}dx.$$

Using integration by parts, we compute

$$\begin{aligned}\int_0^{\infty} xe^{-x}dx = \lim_{b\to\infty}\int_1^b xe^{-x}dx &= \lim_{b\to\infty}\left(\left[-xe^{-x}\right]_1^b + \int_1^b e^{-x}dx\right)\\ &= \lim_{b\to\infty}\left[-xe^{-x} - e^{-x}\right]_1^b\\ &= 2e^{-1} < \infty.\end{aligned}$$

By the integral test we conclude that $\sum_{n=1}^{\infty} ne^{-n^{3/2}}$ converges.

4. (20 points)

Find the arc-length of the parametric curve

$$x = 3\cos t - \cos 3t\,,\ y = 3\sin t - \sin 3t\,,\ 0 \le t \le \pi\,.$$

Answer:

We have

$$dx/dt = -3(\sin t - \sin 3t) \qquad \text{and} \qquad dy/dt = 3(\cos t - \cos 3t).$$

Therefore

$$\begin{aligned}
(ds/dt)^2 &= (dx/dt)^2 + (dy/dt)^2 \\
&= 9(\sin t - \sin 3t)^2 + 9(\cos t - \cos 3t)^2 \\
&= 9(\sin^2 t - 2\sin t \sin 3t + \sin^2 3t + \cos^2 t - 2\cos t\cos 3t + \cos^2 3t) \\
&= 9(2 - 2\cos 2t) \\
&\qquad \text{since } \cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta \\
&= 36\frac{1-\cos 2t}{2} \\
&= 36\sin^2 t,
\end{aligned}$$

so

$$\frac{ds}{dt} = 6|\sin t|.$$

By the arc length formula, we have

$$\begin{aligned}
L &= \int_0^\pi ds \\
&= \int_0^\pi 6|\sin t|dt \\
&= 6\int_0^\pi \sin t dt \\
&= -6\cos t\Big|_0^\pi \\
&= 12.
\end{aligned}$$

5. (20 points)

(a) Compute the area of surface of revolution obtained by rotating the curve $y = x^3$, for $0 \le x \le 1$, about the x-axis.

Answer:

(b) Do the same for the curve $y = |x|$, for $-1 \le x \le 1$.

Answer:

(a)

$$\begin{aligned}
A &= 2\pi \int_0^1 y\sqrt{1 + (dy/dx)^2}\,dx \\
&= 2\pi \int_0^1 x^3\sqrt{1 + 9x^4}\,dx \qquad\qquad (u = 1 + 9x^4,\ du = 36x^3\,dx) \\
&= \frac{2\pi}{36}\int_1^{10} \sqrt{u}\,du \\
&= \frac{2\pi}{36}\cdot\frac{2}{3}u^{3/2}\Big|_1^{10} \;=\; \frac{\pi}{27}\left(10^{3/2} - 1\right) \;=\; \frac{\pi}{27}\left(10\sqrt{10} - 1\right)
\end{aligned}$$

(b) We have

$$\frac{dy}{dx} = \begin{cases} -1 & \text{for } x < 0 \\ 1 & \text{for } x > 0 \end{cases}$$

so

$$\begin{aligned}
A &= 2\pi\int_{-1}^0 (-x)\sqrt{1 + (-1)^2}\,dx + 2\pi\int_0^1 x\sqrt{1 + 1^2}\,dx \\
&= 2\pi\sqrt{2}\left(\left[-\frac{x^2}{2}\right]_{-1}^0 + \left[\frac{x^2}{2}\right]_0^1\right) \\
&= 2\pi\sqrt{2}\left(\frac{1}{2} + \frac{1}{2}\right) = 2\pi\sqrt{2}
\end{aligned}$$