Math 162: Calculus IIA

Second Midterm Exam ANSWERS
November 5, 2015

1. (20 points) The following are convergent series. Find the sum of the series (You don't need to justify the convergence of the series, but you should clearly show how you got the answer).

(a)
$$\sum_{n=1}^{\infty} \frac{1+3^n}{2^{2n}}$$

Answer:

(b)
$$\sum_{n=1}^{\infty} \frac{18}{n(n+3)}$$

Answer:

Solution: (a)

$$\frac{1+3^n}{2^{2n}} = \left(\frac{1}{4}\right)^n + \left(\frac{3}{4}\right)^n.$$

From

$$\sum_{n=1}^{\infty} \left(\frac{1}{4}\right)^n = \frac{1}{4} \sum_{n=1}^{\infty} \left(\frac{1}{4}\right)^{(n-1)} = \frac{1}{4} \frac{1}{1-\frac{1}{4}} = \frac{1}{3}$$

and

$$\sum_{n=1}^{\infty} \left(\frac{3}{4}\right)^n = \frac{3}{4} \sum_{n=1}^{\infty} \left(\frac{3}{4}\right)^{(n-1)} = \frac{3}{4} \frac{1}{1-\frac{3}{4}} = 3,$$

$$\sum_{n=1}^{\infty} \frac{1+3^n}{2^{2n}} = \sum_{n=1}^{\infty} \left(\frac{1}{4}\right)^n + \sum_{n=1}^{\infty} \left(\frac{3}{4}\right)^n = \frac{1}{3} + 3 = \frac{10}{3}.$$

(b) From

$$\frac{3}{n(n+3)} = \frac{1}{n} - \frac{1}{n+3},$$

$$\sum_{n=1}^{\infty} \frac{3}{n(n+3)} = \left(1 - \frac{1}{4}\right) + \left(\frac{1}{2} - \frac{1}{5}\right) + \left(\frac{1}{3} - \frac{1}{6}\right) + \left(\frac{1}{4} - \frac{1}{7}\right) + \dots$$

$$= 1 + \frac{1}{2} + \frac{1}{3} = \frac{11}{6}.$$

So,

$$\sum_{n=1}^{\infty} \frac{18}{n(n+3)} = 6 \cdot \frac{11}{6} = 11.$$

2. (20 points) (a) Find the area of the region that is inside the circle given by $r = 2\cos\theta$ and outside the circle given by $r = \sqrt{2}$ (both equations are in polar coordinates).

Answer:

(b) Compute the equation of the tangent line to the circle $r = 2\cos\theta$ at the point $\left(\frac{3}{2}, \frac{\sqrt{3}}{2}\right)$.

Answer:

Solution: (a)

First we find the values of θ at which the two circles intersect. Setting $2\cos\theta = \sqrt{2}$ we find that $\cos\theta = \frac{\sqrt{2}}{2}$ and so $\theta = \frac{\pi}{4}$ or $-\frac{\pi}{4}$. So

$$A = \int_{-\pi/4}^{\pi/4} \frac{1}{2}\left((2\cos\theta)^2 - \sqrt{2}^2\right) d\theta = \int_{-\pi/4}^{\pi/4} 1 + \cos(2\theta) - 1 \, d\theta = \frac{\sin(2\theta)}{2}\Big|_{-\pi/4}^{\pi/4} = 1$$

(b) First note that the point $\left(\frac{3}{2}, \frac{\sqrt{3}}{2}\right)$ occurs at $\theta = \pi/6$. Using the formula for $\frac{dy}{dx}$ in polar coordinates,

$$\frac{dy}{dx} = \frac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta} = \frac{-2\sin^2\theta + 2\cos^2\theta}{-4\sin\theta\cos\theta}.$$

Evaluating this at $\theta = \pi/6$ gives $\frac{dy}{dx} = -\frac{1}{\sqrt{3}}$. So the line is given by the equation

$$y = \frac{\sqrt{3}}{2} - \frac{1}{\sqrt{3}}\left(x - \frac{3}{2}\right) = \sqrt{3} - \frac{x}{\sqrt{3}}.$$

3. (20 points) Determine whether the following series converge or diverge. Justify your answers, making sure to name the convergence test(s) that you are using.

(a)
$$\sum_{n=2}^{\infty} \frac{\ln(n)}{n^2} = \frac{\ln(2)}{4} + \frac{\ln(3)}{9} + \frac{\ln(4)}{16} + \cdots$$

Answer:

(b)
$$\sum_{n=2}^{\infty} \frac{n^3 - n^2}{n^3 + 1} = \frac{4}{9} + \frac{18}{28} + \frac{48}{65} + \cdots$$

Answer:

Solution: (a) We use the integral test: $f(x) = \frac{\ln(x)}{x^2}$ is continuous, positive and decreasing for $x \geq 2$. Using integration by parts, we compute

$$\int_2^{\infty} \frac{\ln(x)}{x^2} dx = \lim_{b \to \infty} \int_2^b \frac{\ln(x)}{x^2} dx = \lim_{b \to \infty} \left(\left[-\frac{\ln(x)}{x} \right]_2^b + \int_2^b \frac{dx}{x^2} \right)$$

$$= \lim_{b \to \infty} \left[-\frac{\ln(x) + 1}{x} \right]_2^b = \lim_{b \to \infty} \left(\frac{\ln(2) + 1}{2} - \frac{\ln(b) + 1}{b} \right)$$

$$= \frac{\ln(2) + 1}{2} < \infty.$$

By the integral test we conclude that $\sum_{n=1}^{\infty} \frac{\ln(n)}{n^2}$ converges.

(b) Notice that

$$\lim_{n \to \infty} \frac{n^3 - n^2}{n^3 + 1} = \lim_{n \to \infty} \frac{1 - \frac{1}{n}}{1 + \frac{1}{n^3}} = 1,$$

so that $\lim_{n \to \infty} \frac{n^3 - n^2}{n^3 + 1} = 1 \neq 0$, and by the divergence test we conclude that $\sum_{n=2}^{\infty} \frac{n^3 - n^2}{n^3 + 1}$ diverges.

4. (20 points) Consider the curve

$$f(x) = \frac{x^4}{4} + \frac{1}{8x^2}.$$

(a) Calculate the arc length function $s(t)$ starting at $x = 1$, that computes the length of the curve from $(1, f(1))$ to $(t, f(t))$.

Answer:

(b) Calculate the arc length from $x = 2$ to $x = 4$.

Answer:

Solution: (a) Since

$$f'(x) = x^3 - \frac{1}{4x^3},$$

the arc length function is given by

$$
\begin{aligned}
s(t) &= \int_1^t \sqrt{1 + \left(x^3 - \frac{1}{4x^3}\right)^2}\, dx = \int_1^t \sqrt{\left(x^3 + \frac{1}{4x^3}\right)^2}\, dx \\
&= \int_1^t \left(x^3 + \frac{1}{4x^3}\right) dx = \left.\left(\frac{x^4}{4} - \frac{1}{8x^2}\right)\right|_1^t \\
&= \left(\frac{t^4}{4} - \frac{1}{8t^2}\right) - \left(\frac{1}{4} - \frac{1}{8}\right) = \frac{t^4}{4} - \frac{1}{8t^2} - \frac{1}{8}
\end{aligned}
$$

(b) By the definition of the arc length function, $s(4)$ is the arc length from $t = 1$ to $t = 4$ and $s(2)$ is the arc length from $t = 1$ to $t = 4$, so the arc length from $t = 2$ to $t = 4$ is

$$s(4) - s(2) = \frac{4^4 - 2^4}{4} - \left(\frac{1}{8 \cdot 4^2} - \frac{1}{8 \cdot 2^2}\right) = 60 + \frac{3}{128}.$$

5. (20 points)

(a) Compute the area of surface of revolution obtained by rotating the curve $y = \sqrt{9 - x^2}$ around the x-axis.

Answer:

(b) Do the same for the curve $y = 1 - |x|$, $-1 \leq x \leq 1$.

Answer:

Solution:

(a)

$$
\begin{aligned}
A &= 2\pi \int_{-3}^3 y\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = 2\pi \int_{-3}^3 \sqrt{9 - x^2}\sqrt{1 + \frac{x^2}{9 - x^2}}\, dx \\
&= 2\pi \int_{-3}^3 3\, dx = 36\pi
\end{aligned}
$$

(b)

$$
\begin{aligned}
A &= 2\pi \int_{-1}^{0} (1+x)\sqrt{1+1}\,dx + 2\pi \int_{0}^{1} (1-x)\sqrt{1+1}\,dx \\
&= 2\pi\sqrt{2}\left(\left[x + \frac{x^2}{2}\right]_{-1}^{0} + \left[x - \frac{x^2}{2}\right]_{0}^{1}\right) \\
&= 2\pi\sqrt{2}\left(\frac{1}{2} + \frac{1}{2}\right) = 2\pi\sqrt{2}
\end{aligned}
$$