Math 162: Calculus IIA

Second Midterm Exam ANSWERS

November 14, 2013

1. (20 points) Consider the curve

$$f(x) = \frac{x^4}{16} + \frac{1}{2x^2}.$$

(a) Calculate the arc length function $s(t)$ starting at $x = 1$, that computes the length of the curve from $(1, f(1))$ to $(t, f(t))$.

(b) Calculate the arc length from $x = 2$ to $x = 4$.

Solution: (a) Since

$$f'(x) = \frac{x^3}{4} - \frac{1}{x^3},$$

the arc length function is given by

$$\begin{aligned} s(t) &= \int_1^t \sqrt{1 + \left(\frac{x^3}{4} - \frac{1}{x^3}\right)^2}\, dx \\ &= \int_1^t \sqrt{\left(\frac{x^3}{4} + \frac{1}{x^3}\right)^2}\, dx \\ &= \int_1^t \frac{x^3}{4} + \frac{1}{x^3} dx \\ &= \frac{t^4}{16} - \frac{1}{2t^2} + \frac{7}{16}. \end{aligned}$$

(b) By the definition of the arc length function, $s(4)$ is the arc length from $t = 1$ to $t = 4$ and $s(2)$ is the arc length from $t = 1$ to $t = 4$, so the arc length from $t = 2$ to $t = 4$ is

$$s(4) - s(2) = 15 + \frac{3}{32}.$$

2. (20 points)

(a) Find the area of the surface of revolution obtained by rotating the curve $y = x^2$, for $0 \leq x \leq 2$, about the y-axis.

(b) Find the area of the surface of revolution obtained by rotating the curve $x = 1 + |y|$, for $-1 \leq y \leq 1$, about the y-axis.

Solution: (a) We compute

$$A = \int_0^2 2\pi x\sqrt{1 + (dy/dx)^2}dx = 2\pi \int_0^2 x\sqrt{1 + 4x^2}dx.$$

Setting $u = 1 + 4x^2$, we have $du = 8xdx$, so that $xdx = du/8$. Also when $x = 0$, we have $u = 1$, and when $x = 2$, $u = 17$. This gives

$$A = \frac{\pi}{4}\int_1^{17} \sqrt{u}du = \frac{\pi}{4}\left[\frac{2}{3}u^{3/2}\right]_1^{17} = \frac{\pi}{6}(17^{3/2} - 1).$$

Solution: (b) Recall that $|y| = \begin{cases} y & \text{if } y \geq 0, \\ -y & \text{if } y < 0, \end{cases}$ and therefore $\dfrac{dx}{dy} = \begin{cases} 1 & \text{if } y \geq 0, \\ -1 & \text{if } y < 0. \end{cases}$ We now compute

$$\begin{aligned} A &= \int_{-1}^1 2\pi(1 + |y|)\sqrt{1 + (dx/dy)^2}dy \\ &= 2\pi \int_{-1}^0 (1 - y)\sqrt{1 + (-1)^2}dy + 2\pi \int_0^1 (1 + y)\sqrt{1 + (1)^2}dy \\ &= 2\sqrt{2}\pi\left[y - \frac{y^2}{2}\right]_{-1}^0 + 2\sqrt{2}\pi\left[y + \frac{y^2}{2}\right]_0^1 \\ &= 2\sqrt{2}\pi\left(-\left(-1 - \frac{(-1)^2}{2}\right)\right) + 2\sqrt{2}\pi\left(1 + \frac{1^2}{2}\right) = 6\sqrt{2}\pi. \end{aligned}$$

3. (20 points) Determine whether the following series converge or diverge. Justify your answers, making sure to name the convergence test(s) that you are using.

(a)

$$\sum_{n=1}^{\infty} \frac{3^n + 1}{2^n - 1} = 4 + \frac{10}{3} + \frac{28}{7} + \frac{82}{15} + \cdots$$

(b)

$$\sum_{n=2}^{\infty} \frac{1}{n\ln(n)^3} = \frac{1}{2\ln(2)^3} + \frac{1}{3\ln(3)^3} + \frac{1}{4\ln(4)^3} + \cdots$$

Solution: (a) We will show that the terms in the series do not tend to zero and n tends to infinity. We have

$$\frac{3^n + 1}{2^n - 1} > \frac{3^n}{2^n} = \left(\frac{3}{2}\right)^n$$

so

$$\lim_{n\to\infty} \frac{3^n+1}{2^{n-1}} > \lim_{n\to\infty} \left(\frac{3}{2}\right)^n = \infty.$$

hence the series diverges by the Divergence Test.

(b) We will use the Integral Test and show that the improper integral

$$\int_2^\infty \frac{dx}{x\ln(x)^3}$$

converges. We will use the substitution

$$u = \ln(x) \qquad du = \frac{dx}{x},$$

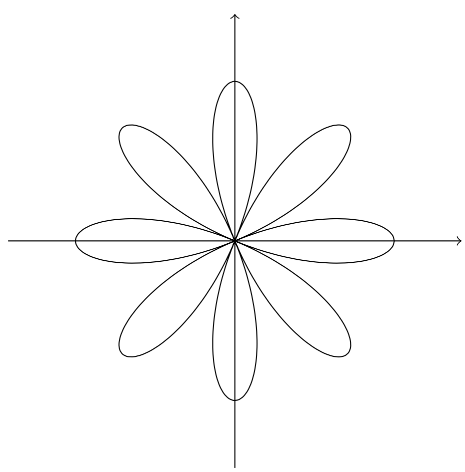
which gives

$$\int_2^\infty \frac{dx}{x\ln(x)^3} = \int_{\ln 2}^\infty \frac{du}{u^3} = -\frac{1}{2u^2}\bigg|_{\ln 2}^\infty = \frac{1}{2(\ln 2)^2}.$$

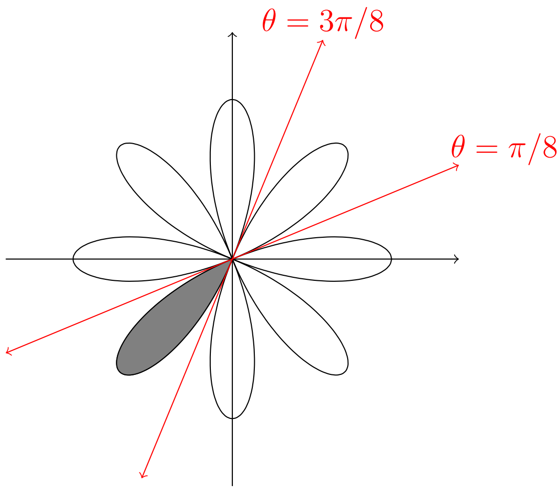
This the integral; converges, so the series does.

4. (20 points)

(a) Find the area of **one petal** of the polar rose $r = 2\cos(4\theta)$ pictured below.



Solution: We need to find consecutive zeros of $r = 2\cos(4\theta)$. These will give the limits of integration, because the petal will close for those θ values. If $0 = 2\cos(4\theta)$, then $4\theta = \pi/2, 3\pi/2$, so $\theta = \pi/8, 3\pi/8$ are the limits of integration.



$$
\begin{aligned}
\text{Area} &= \frac{1}{2}\int_{\pi/8}^{3\pi/8} 4\cos^2(4\theta)d\theta \\
&= 2\int_{\pi/8}^{3\pi/8} \frac{1+\cos(8\theta)}{2}d\theta \\
&= \theta + \frac{\sin(8\theta)}{8}\Big|_{\pi/8}^{3\pi/8} \\
&= \pi/4
\end{aligned}
$$

Other correct integrals: $\frac{1}{8\frac{1}{2}}\int_0^{2\pi} 4\cos^2(4\theta)d\theta$, $2\frac{1}{2}\int_0^{\pi/8} 4\cos^2(4\theta)d\theta$, $\frac{1}{2}\int_{3\pi/8}^{5\pi/8} 4\cos^2(4\theta)d\theta$, etc.

(b) The parametric curve given by $x = 4t^3 - 3t$, $y = t^2 + 1$ intersects the y-axis at 3 different values of t. What are the **equations of the tangent lines** to the curve at each of these points?

Solution: Solve $x = 4t^3 - 3t = 0$ to get $t = 0, \pm\frac{\sqrt{3}}{2}$. We have

$$\frac{dy}{dx} = \frac{2t}{12t^2 - 3}$$

At $t = 0$, the tangent is horizontal with y-intercept $y = 1$, so we get $y = 1$. At $t = \frac{\sqrt{3}}{2}$, at $(x, y) = (0, 1 + 3/4)$, the tangent has slope $\frac{\sqrt{3}}{6}$. At $t = -\frac{\sqrt{3}}{2}$, also at $(x, y) = (0, 1 + 3/4)$, the tangent has slope $-\frac{\sqrt{3}}{6}$. So the two lines are

$$y = \pm\frac{\sqrt{3}}{6}x + 7/4.$$

5. (20 points) Find the sum of each of the following series.

(a)

$$\sum_{n=2}^{\infty} \frac{2}{n^2-1} = \frac{2}{3} + \frac{2}{8} + \frac{2}{15} + \frac{2}{24} + \frac{2}{35} + \cdots$$

HINT: USE PARTIAL FRACTIONS.

(b)

$$\sum_{n=0}^{\infty} \left(\frac{1}{6+(-1)^n}\right)^n = 1 + \frac{1}{5} + \frac{1}{7^2} + \frac{1}{5^3} + \frac{1}{7^4} + \cdots$$

HINT: CONSIDER THE EVENLY AND ODDLY INDEXED TERMS SEPARATELY.

Solution: (a) Using partial fractions we find that

$$\frac{2}{n^2-1} = \frac{1}{n-1} - \frac{1}{n+1}.$$

Hence we can rewrite the series as

$$\begin{aligned}\sum_{n=2}^{\infty} \frac{2}{n^2-1} &= \sum_{n=2}^{\infty} \left(\frac{1}{n-1} - \frac{1}{n+1}\right) \\ &= \left(\frac{1}{1} - \frac{1}{3}\right) + \left(\frac{1}{2} - \frac{1}{4}\right) + \left(\frac{1}{3} - \frac{1}{5}\right) + \left(\frac{1}{4} - \frac{1}{6}\right) + \left(\frac{1}{5} - \frac{1}{7}\right) + \cdots .\end{aligned}$$

All the negative terms are cancelled by subsequent positive ones, leaving only the first two positive terms. Hence the sum is $3/2$.

(b) Collecting the evenly and oddly indexed terms into separate series, we get

$$\begin{aligned}\sum_{n=0}^{\infty} \left(\frac{1}{6+(-1)^n}\right)^n &= 1 + \frac{1}{5} + \frac{1}{7^2} + \frac{1}{5^3} + \frac{1}{7^4} + \cdots \\ &= \left(1 + \frac{1}{7^2} + \frac{1}{7^4} + \cdots\right) + \left(\frac{1}{5} + \frac{1}{5^3} + \frac{1}{5^5} + \cdots\right) \\ &= \left(1 + \frac{1}{7^2} + \frac{1}{7^4} + \cdots\right) + \frac{1}{5}\left(1 + \frac{1}{5^2} + \frac{1}{5^4} + \cdots\right).\end{aligned}$$

This is the sum of two geometric series, namely

$$\begin{aligned}\sum_{m=0}^{\infty} \frac{1}{49^m} + \frac{1}{5}\sum_{m=0}^{\infty} \frac{1}{25^m} &= \frac{1}{1-(1/49)} + \frac{1}{5}\left(\frac{1}{1-(1/25)}\right) \\ &= \frac{49}{48} + \frac{25}{5\cdot 24} = \frac{5\cdot 49 + 2\cdot 25}{5\cdot 48} = \frac{295}{240} = \frac{59}{48}.\end{aligned}$$